Pre-Calculus Course

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Co-funded by the European Union

Pythagoras
2021-1-RO01-KA220-HED-000032258

Pre – Calculus Course

The main objective *of this module is to prepare better the graduate students from high School to follow a University level module in Calculus and Linear Algebra*

The module apart of these lecture notes is also accompanied by: (a) Self – Evaluation Questions; (b) lecture notes; and (c) simulations/visualizations using GeoGebra (www.geogebra.org)

Knocking Down the Myths about Mathematics

Help you to love and learn math to the highest levels

Associate Professor Konstantinos Petridis

Department of Electronic Engineering, Hellenic Mediterranean University

The Motivation

"changing the stories (or myths) told about mathematics is necessary for changing the way mathematics is done and the way it is taught. We emphasize the need for change to combat the sense of repression often associated with mathematics"

The Outline

- The Myths about Mathematics
- The planned actions within the framework of the iTEM project

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Knocking Down the Myths About Maths

People they do not like Math, mainly because of the way it is taught

The Myths

‘Most of the population perceive mathematics as a fixed body of knowledge long set into final form. Its subject matter is the manipulation of numbers and the proving of geometrical deductions. It is a cold and austere discipline which provides no scope for judgment or creativity’

A set of basically meaningless disconnected procedures that have to be memorized.

Mathematics is only for engineers

“My daughter just does not understand math. I told her, ‘Don’t worry, honey. I was never good at math either.”

the robust myth that mathematical smartness is exemplified in individuals who consistently complete mathematics problems quickly and accurately

Many people seem to believe that, with respect to mathematics, the world consists of two groups: those who are “math people” and those who are not

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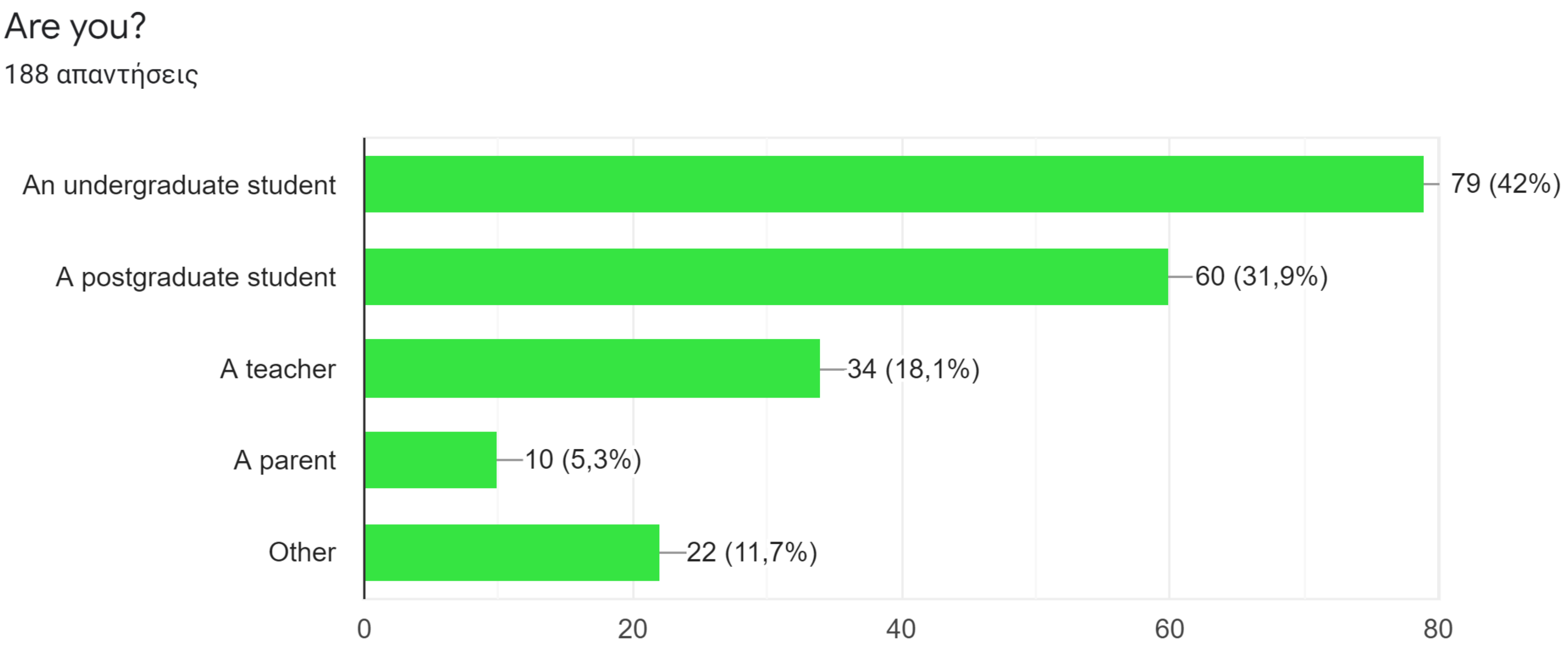
About the survey

- Launched on the 10th of February and closed on the 12th of March 2022.
- Completed by 188 individuals (Students, teachers and other professionals).
- Objective: to mine the myths about Mathematics among the stakeholders.

Participants’ Profile



HMU Survey

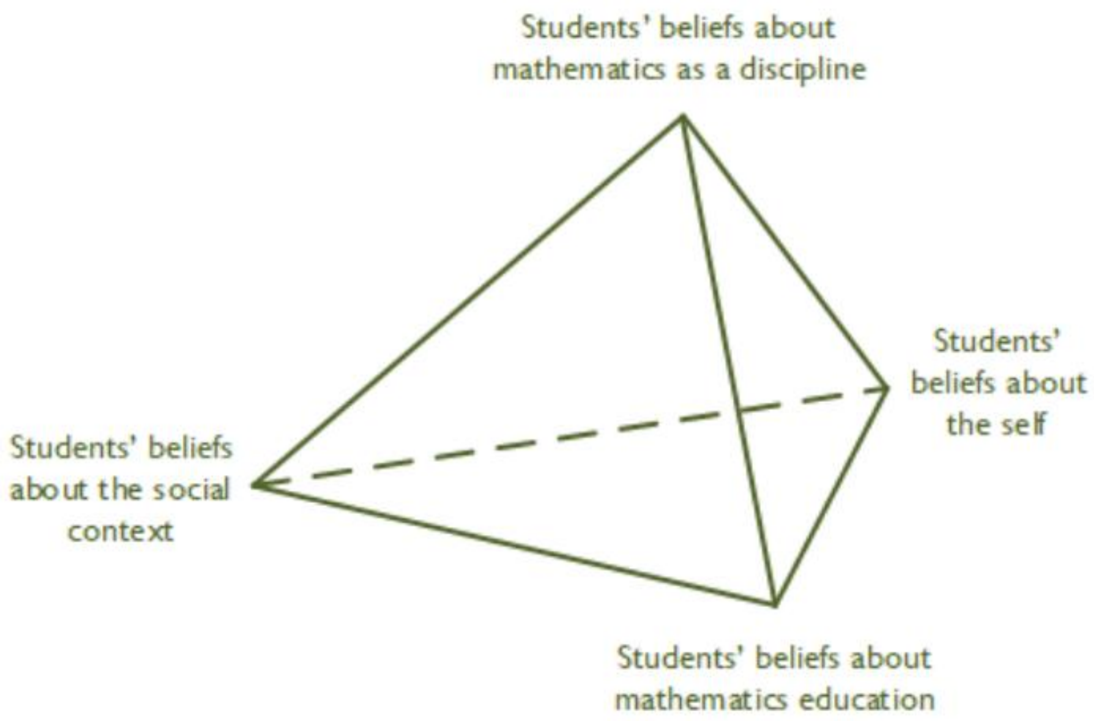
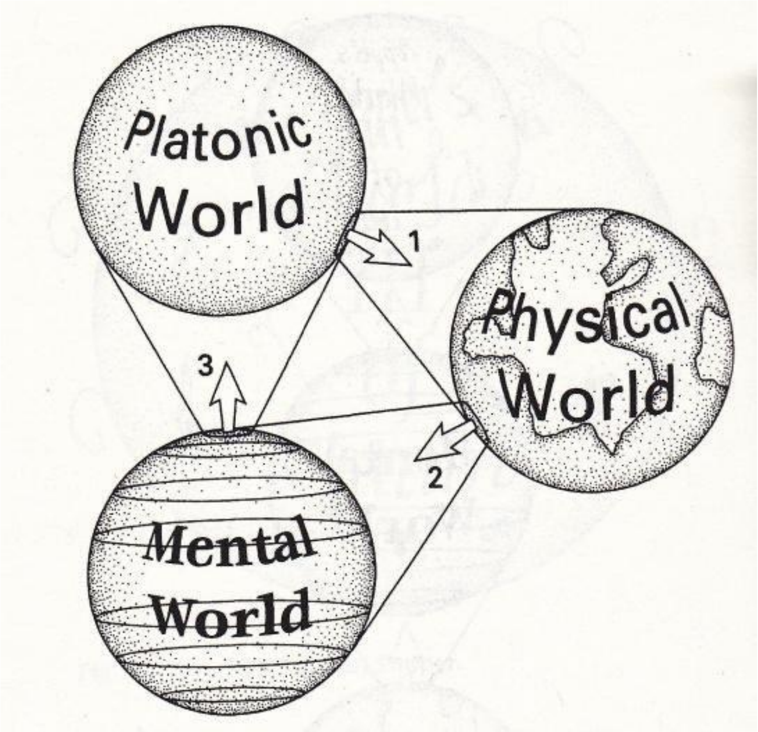


Figure 1. Students' mathematics-related beliefs as made up by a basis of their beliefs about mathematics education, the social context and the self (triangle) [4], and their beliefs about mathematics as a discipline (the point above the triangle resulting in a tetrahedron) [5].

Myth # 1: Mathematical Platonism



- Mathematics exists as a complete structure ‘out there’, just waiting to be discovered (Platonism): Students learn formal skills and algorithms and then they apply those skills in exercises
- **Consequence**: ‘Mathematics’ means memorizing formal algorithms and procedures for abstract symbol-manipulation.
- **Truth**: Mathematics is a human activity; connect mathematics teaching with practices that give them meaning!! Mathematize the world!!!

Myth # 1: Mathematical Platonism

- Mathematics Education should be:
  - **Active:** Engage students n structuring activities rather than starting with the structure of mathematics (active learning).
  - **Cultural**: Create learning environments (classrooms) that facilitates the development of students cognitive powers through their exercise (e.g. the use of technology and visualization tools along learning and teaching).
  - **Historical:** Engaging students in the history of mathematics so to see the role of humans towards their development.
  - **Social**: Mathematics requires social interaction. PBL shows the facilitation of learning through social interaction.

Analytical representation

$$h(t) = 30\sin\left(\frac{\pi(t-18)}{36}\right) + 30$$

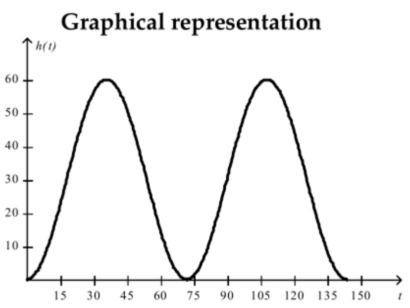
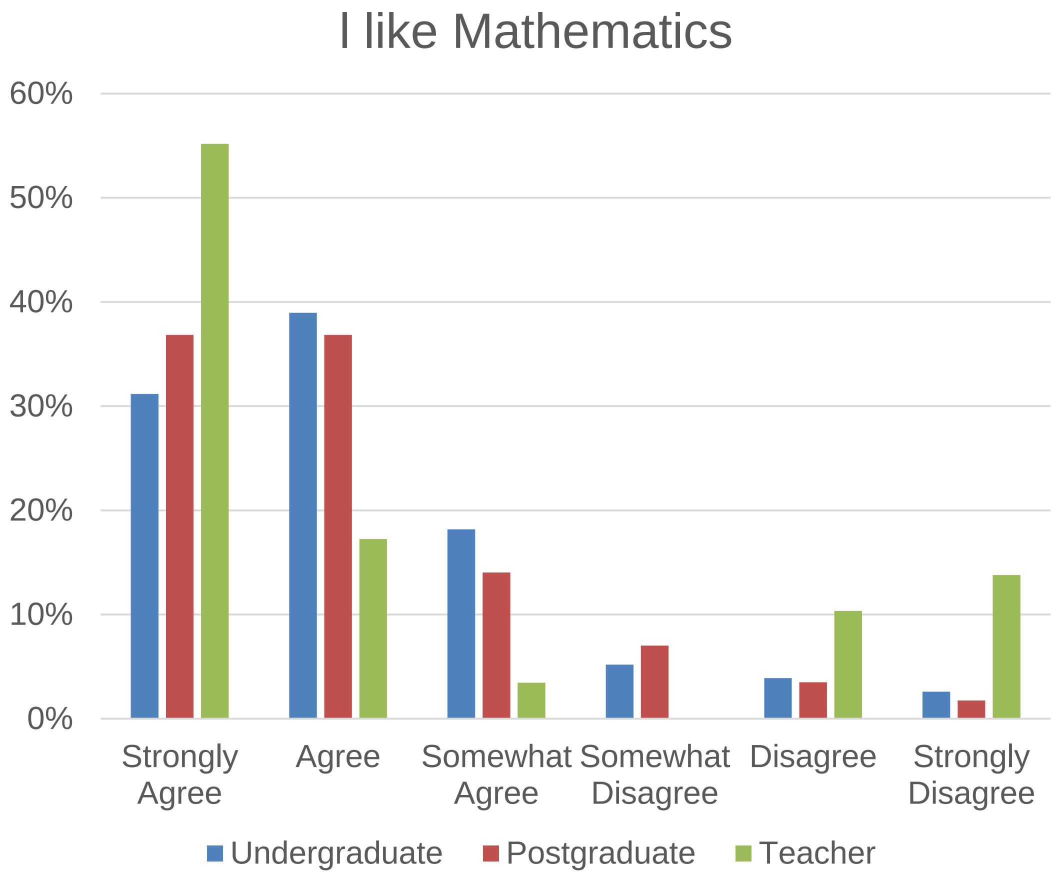


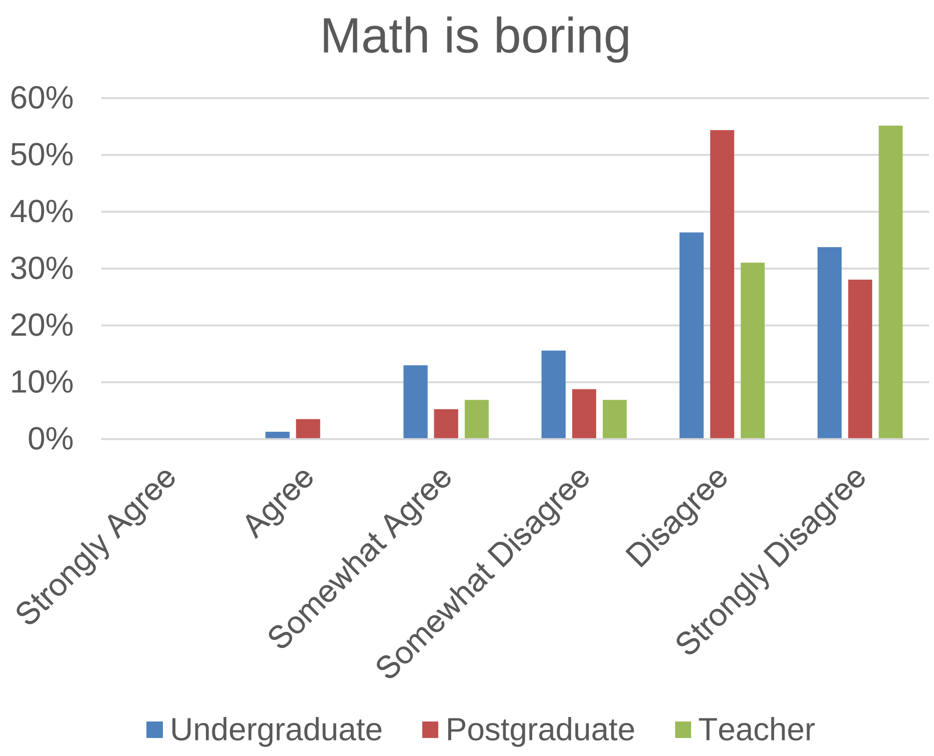
Figure 3. Two representations of the height of a Ferris Wheel as a function of time.

Myth # 1: Mathematical Platonism (I)



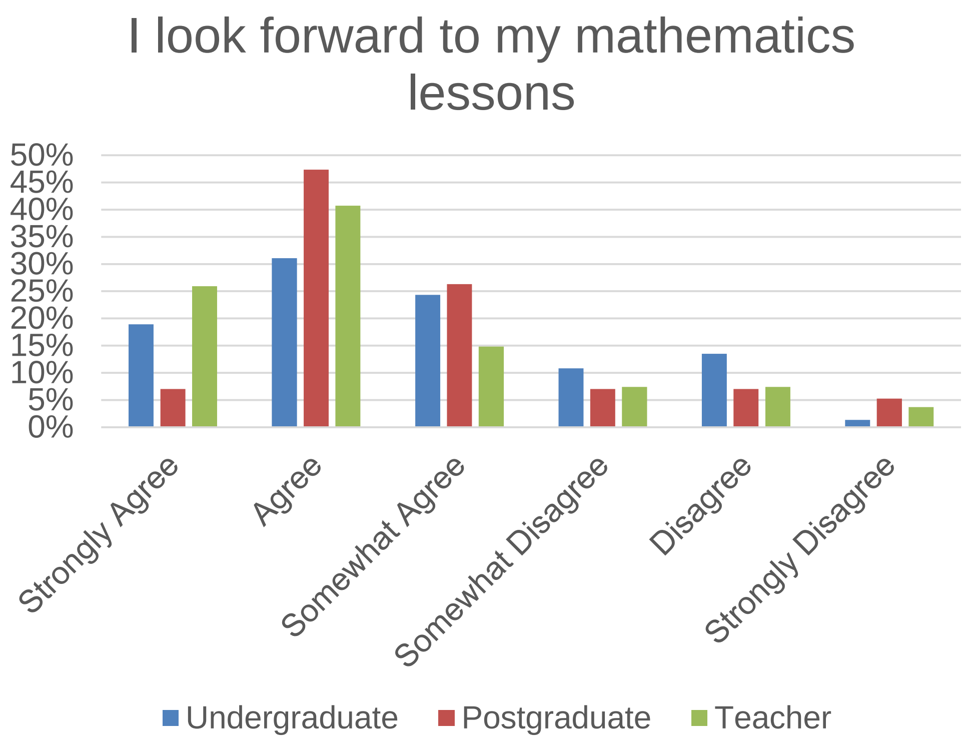
	Undergraduate	Postgraduate	Teachers
Agree	88%	88%	76%
Disagree	12%	12%	24%

Myth # 1: Mathematical Platonism (II)



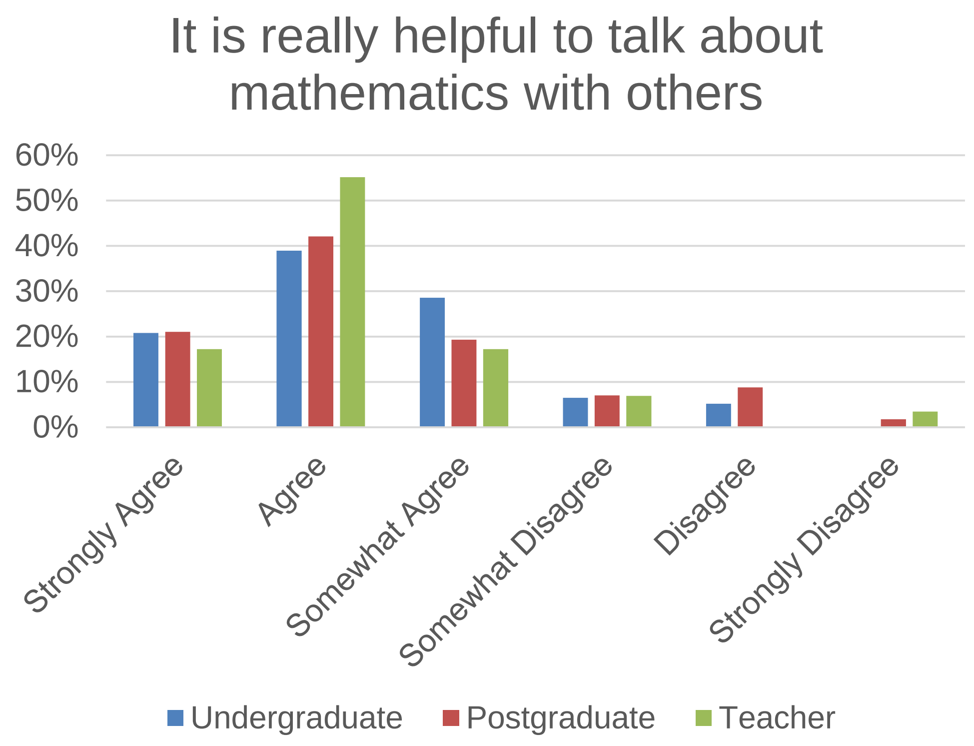
	Undergraduate	**Postgraduate**	**Teachers**
Agree	14%	9%	7%
Disagree	86%	91%	93%

Myth # 1: Mathematical Platonism (III)

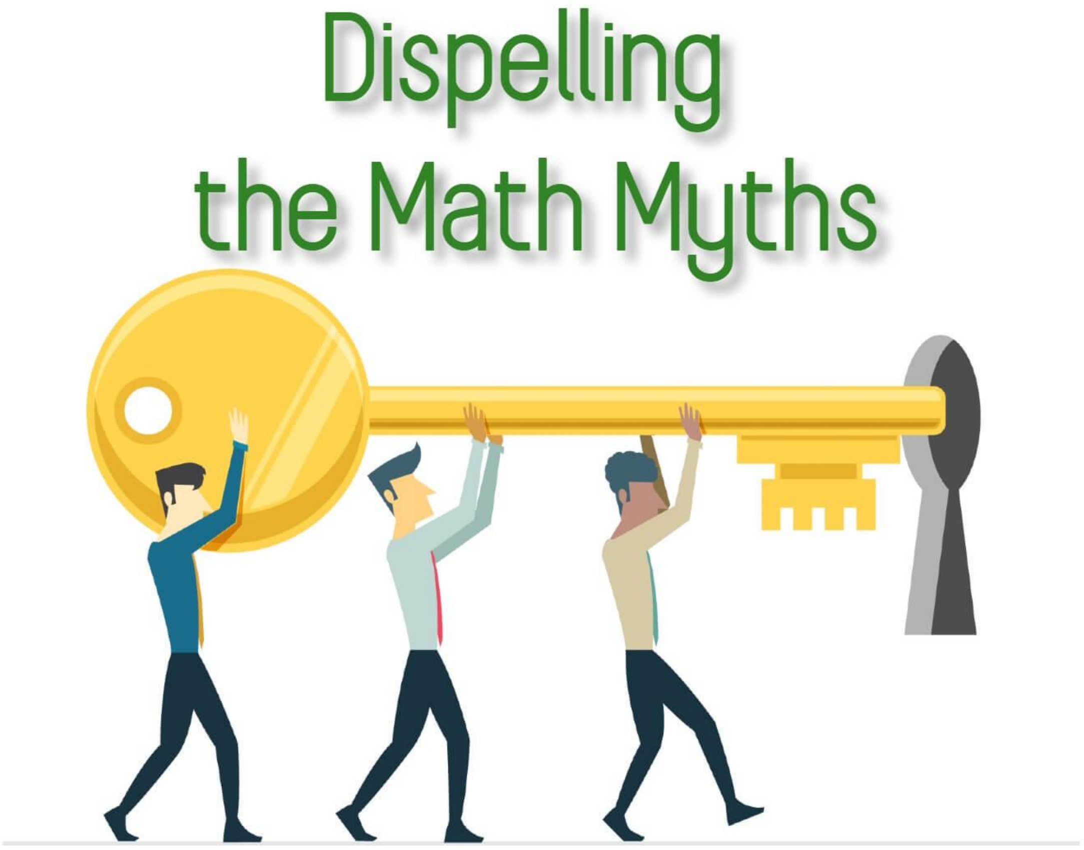


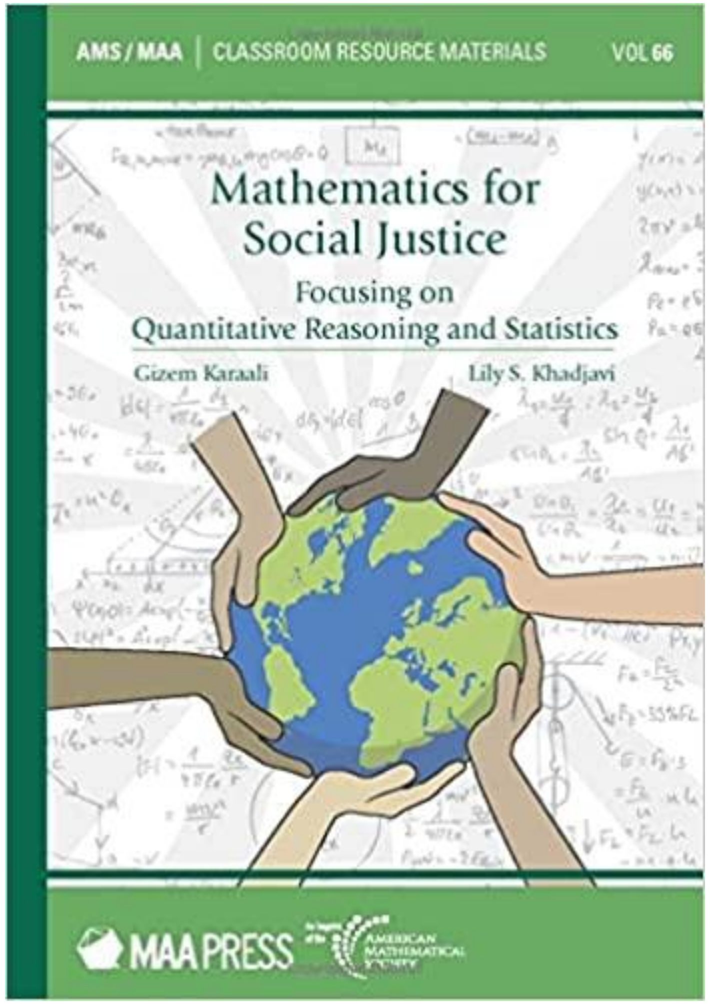
	Undergraduate	**Postgraduate**	**Teachers**
Agree	74%	81%	81%
Disagree	26%	19%	19%

Myth # 1: Mathematical Platonism (IV)



	Undergraduate	**Postgraduate**	**Teachers**
Agree	88%	82%	90%
Disagree	12%	18%	10%





Mathematics are only for Engineers!!

Myth #2: Mathematics is only for engineers

- **Myth #2:** Mathematics is only for engineers!!
- This narrows the role of mathematical literacy in successful democracies !!!
- To show the impact of Mathematics in shaping well informed citizens: engage students in creative Mathematics and connecting those understandings.
- Citizens with **analytical and communicative skills** and understandings necessary to participate in democracy!!

Ontario Grade 12 Advanced Functions

3.3 Solve problems, using a variety of tools and strategies, including problems arising from real-world applications, by reasoning with functions and by applying concepts and procedures involving functions (e.g., by constructing a function model from data, using the model to determine mathematical results, and interpreting and communicating the results within the context of the problem).

Sample Problem. The pressure of a car tire with a slow leak is given in the following table of values:

Time, *t* (min)	**Pressure, *P* (kPa)**
0	400
5	335
10	295
15	255
20	225
25	195
30	170

Use technology to investigate linear, quadratic, and exponential models for the relationship, of the tire pressure and time, and describe how well each model fits the data. Use each model to predict the pressure after 60 min. Which model gives the most realistic answer?

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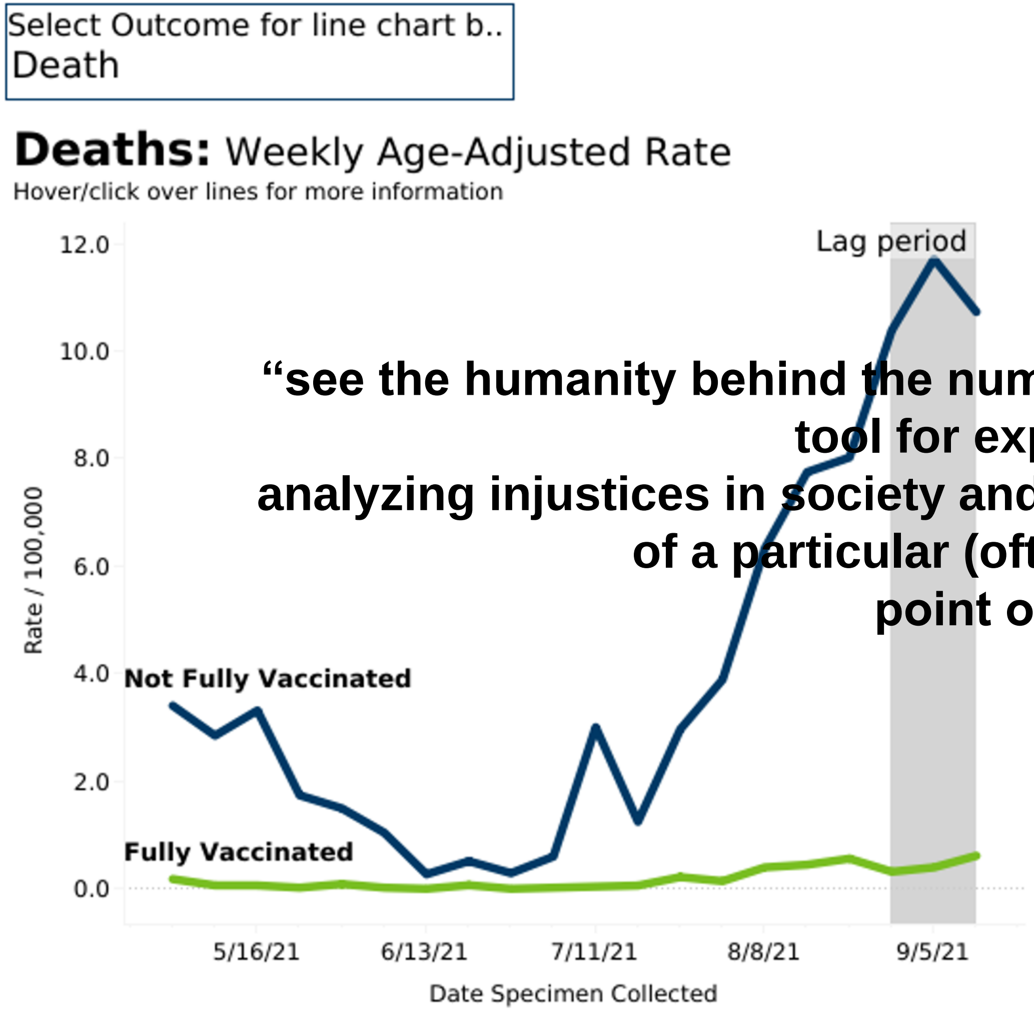
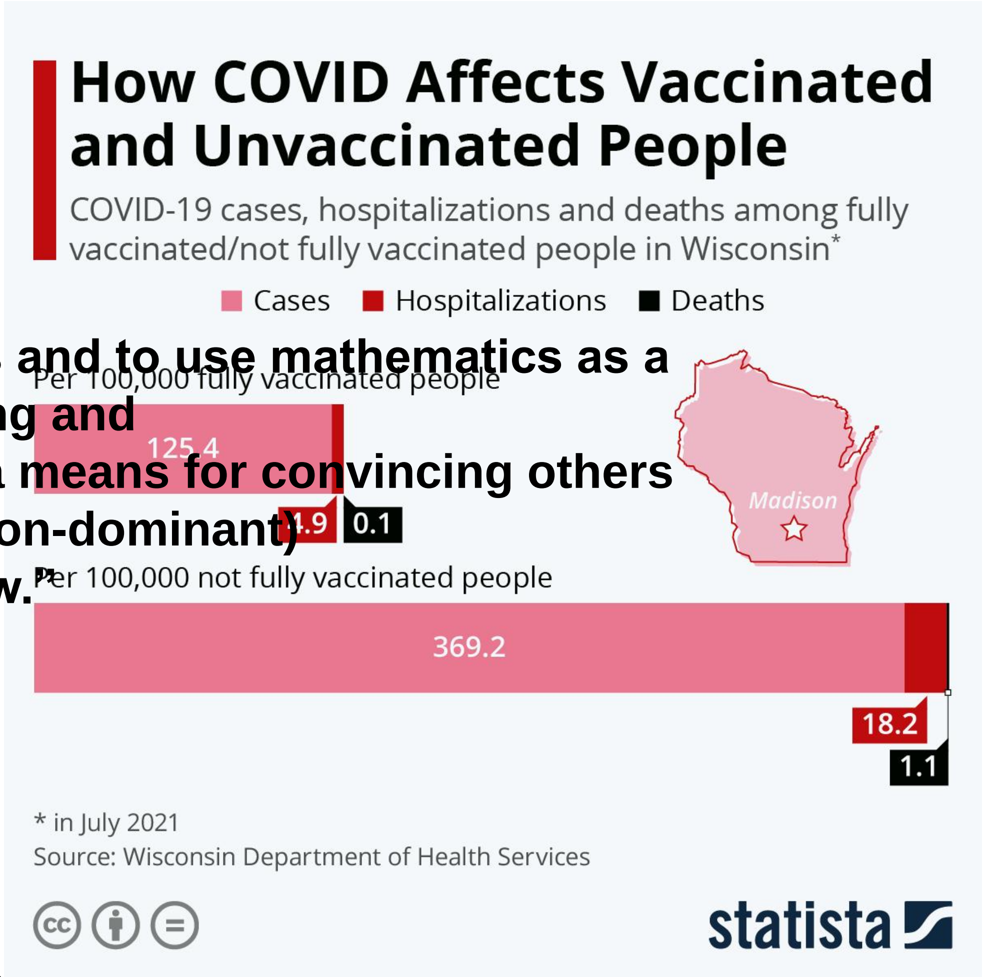
Myth #2: Mathematics is only for engineers





"see the humanity behind the numbers and to use mathematics as a tool for exposing and analyzing injustices in society and as a means for convincing others of a particular (often non-dominant) point of view."

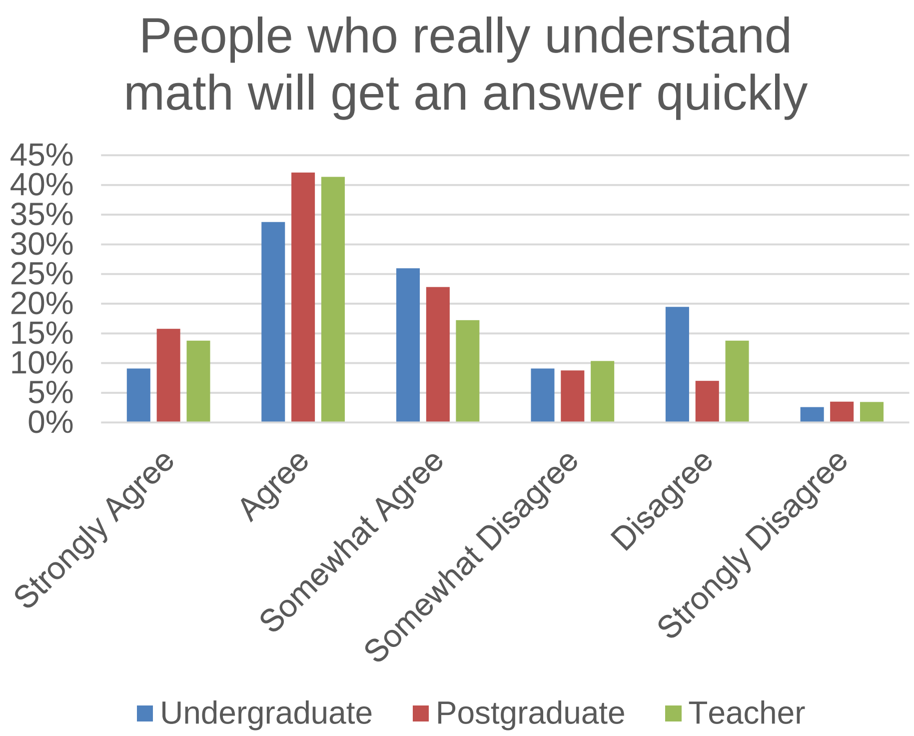
Myth #3: The F1 Myth

- **Myth:** Complete mathematics problems quickly and accurately.

- **Conclusion #1**: Mathematical Learning works against the notion of speed and answer – oriented mathematics.

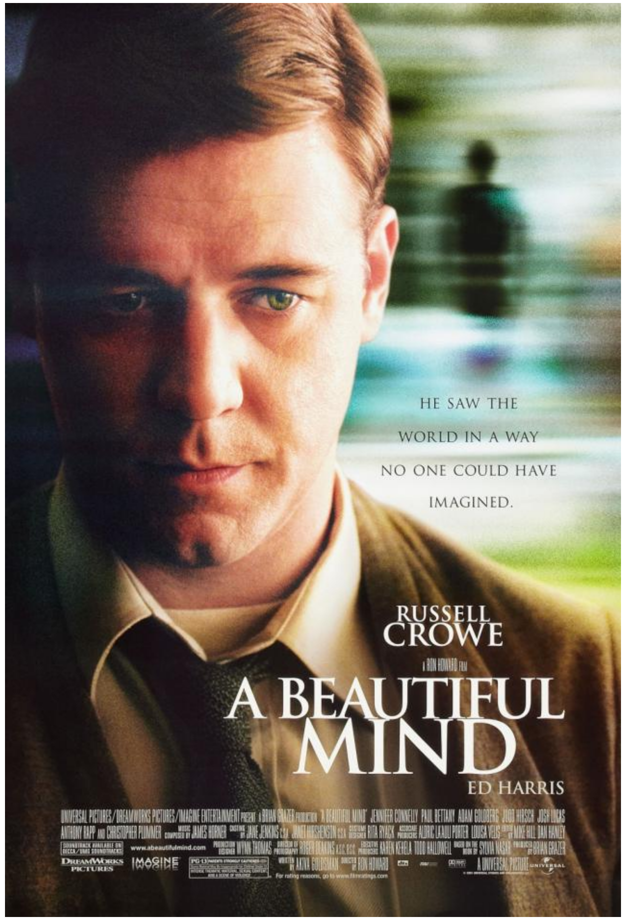
- **Important features in mathematics education**: (a) valuing multiple solution strategies; (b) requiring student explanation and justification; (c) naming students' competence, and (4) valuing collaborative problem solving.

Myth #3: The F1 Myth



	Undergraduate	**Postgraduate**	**Teachers**
Agree	69%	81%	72%
Disagree	31%	19%	28%

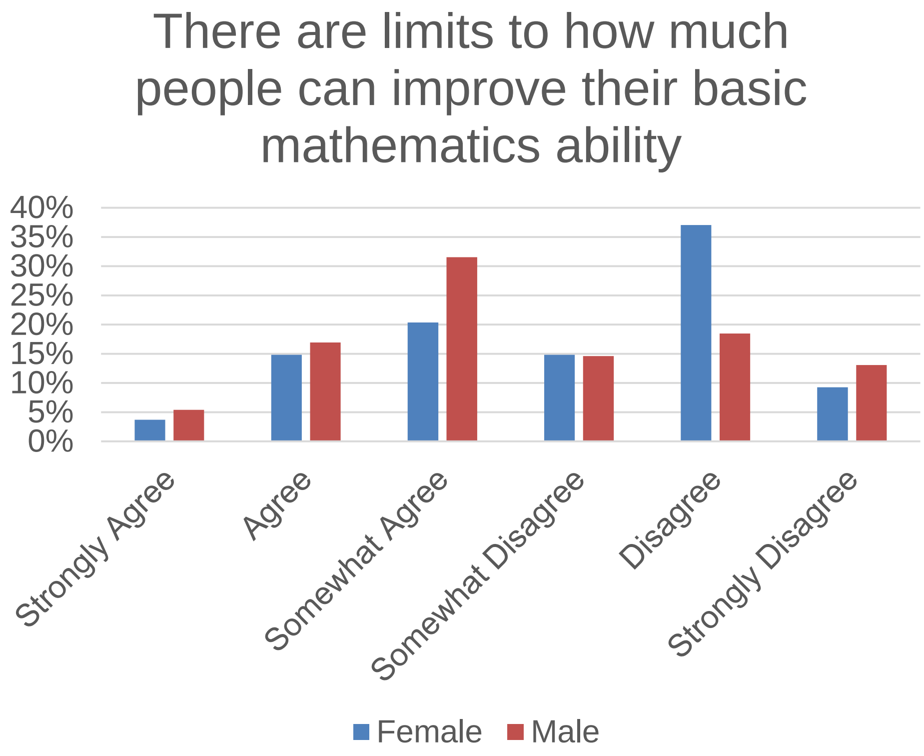
Myth #4: Only Brilliant people are Good at Math



- **Myth:** Math requires raw intellectual talent or ‘brilliance’
- The negative impact of this myth is to place a barrier to **girls and ethic minorities** to follow math-related subjects!!!
- Check the percentages of female scientists in Math related disciplines (natural sciences and engineering)
- The truth: with suitable effort and strategies **every school student can become proficient in mathematics**.
- **Tip**: If, instead, teachers encourage students to engage more deeply with math by incorporating open-ended projects that are contextualized within the world outside the classroom (**Math meets Industry**)

Myth 4: Only Brilliant people are good in Math

Myth #4: Only Brilliant people are Good at Math



	Total	Male	Female
Agree	49%	54%	39%
Disagree	51%	46%	61%

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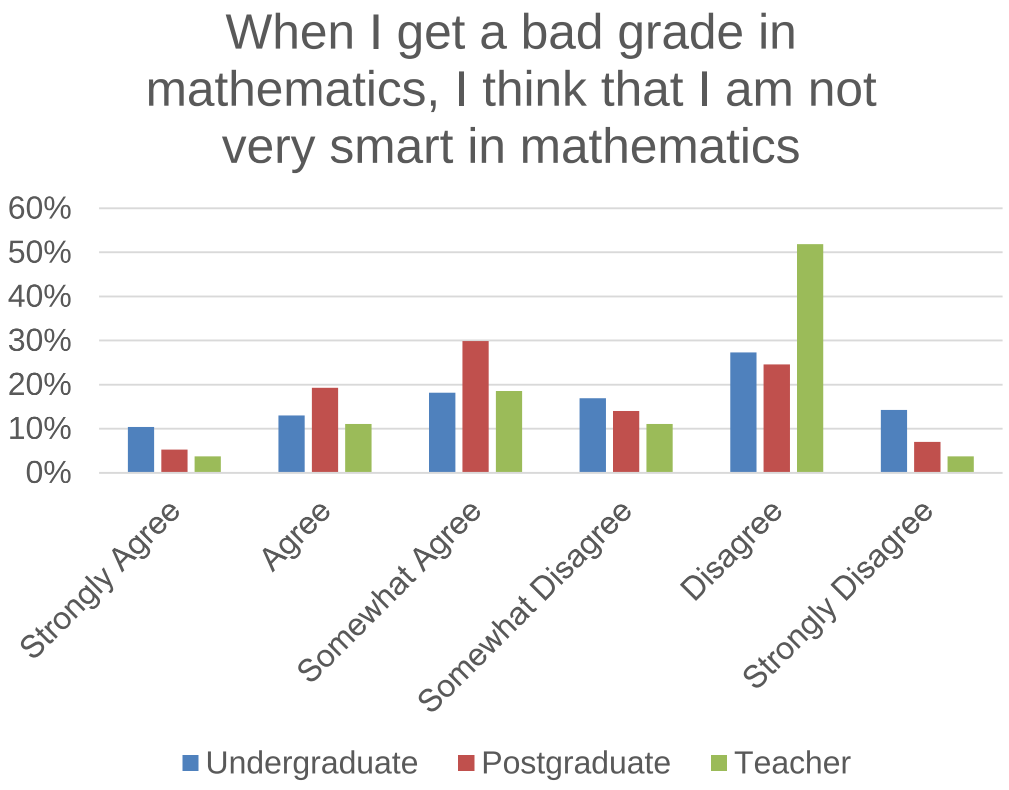
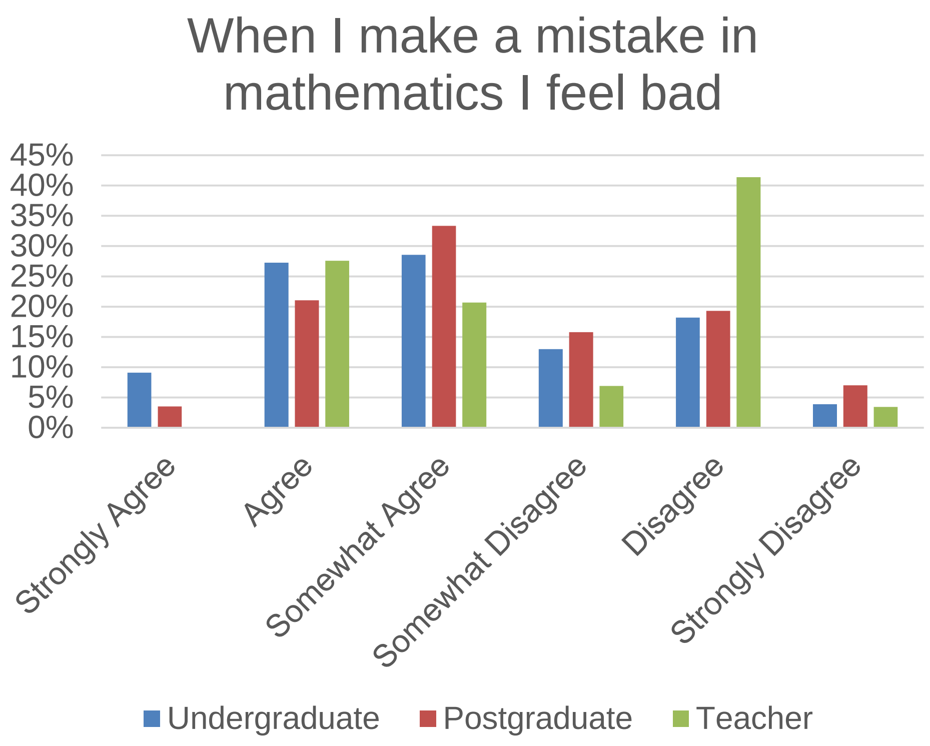
Myth #4: Only Brilliant people are Good at Math



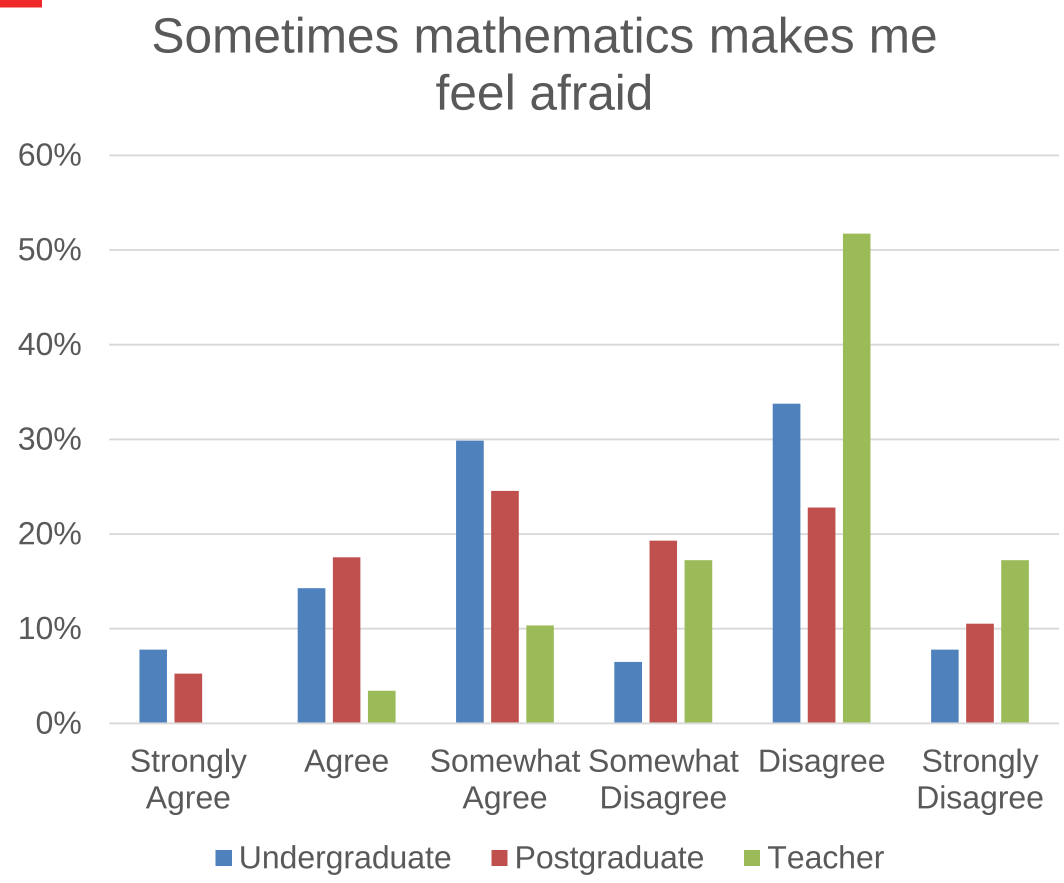
	Undergraduate	Postgraduate	Teachers
Agree	42%	54%	33%
Disagree	58%	46%	67%

Myth #4: Only Brilliant people are Good at Math



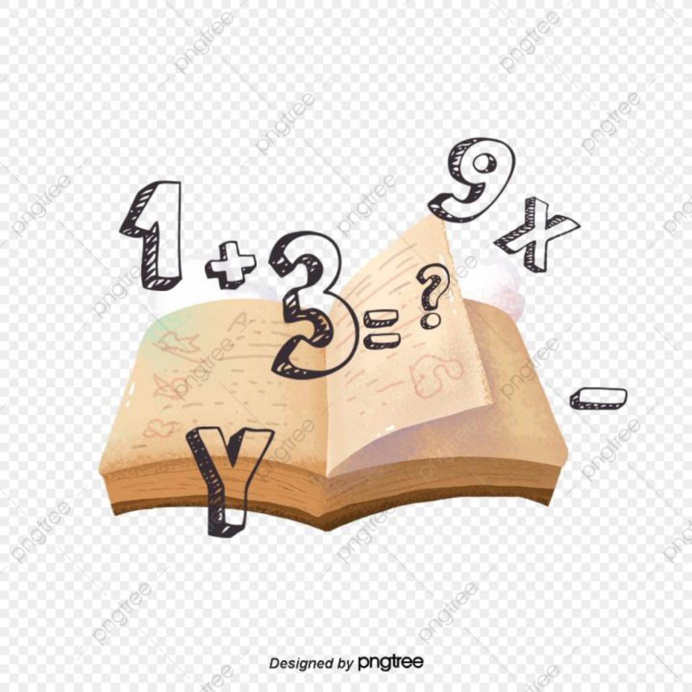
	Undergraduate	Postgraduate	Teachers
Agree	65%	58%	48%
Disagree	35%	42%	52%

Myth #4: Only Brilliant people are Good at Math



	Undergraduate	**Postgraduate**	**Teachers**
Agree	52%	47%	14%
Disagree	48%	53%	86%

Myth #5: Misconceptions



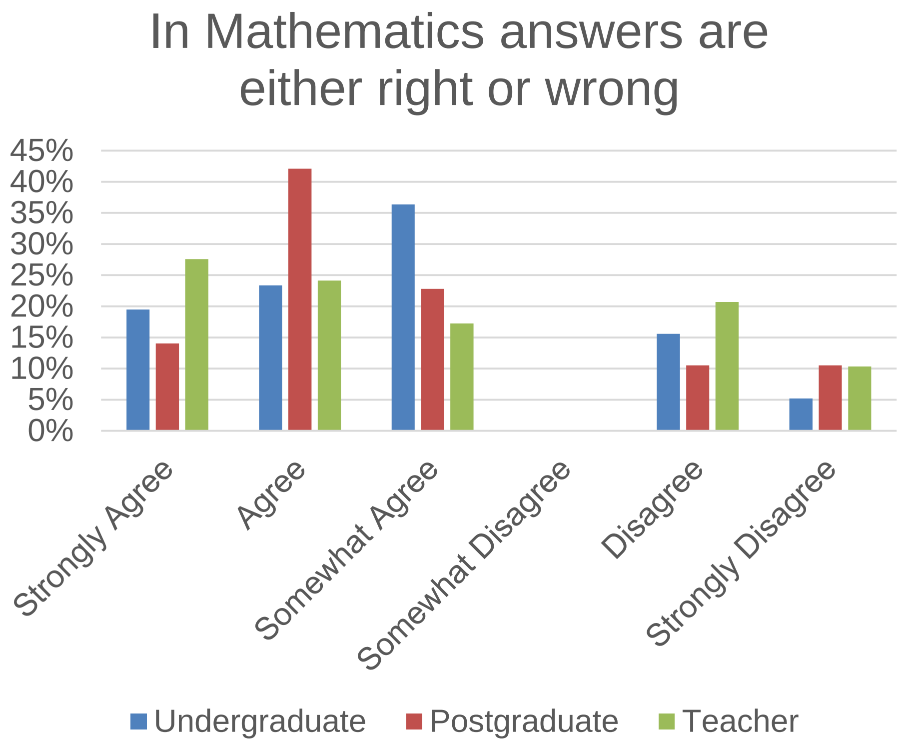
- Students' conception of mathematics **as a set of basically meaningless disconnected procedures that have to be memorized.**

- According to Schoenfeld et al. 'conception of the nature of mathematics' other misconceptions include:

  - ❑ Mathematics problems have one and only one right answer.
  - ❑ There is only one correct way to solve any mathematics problem.
  - ❑ Ordinary students can memorize and not understand mathematics.
  - ❑ Mathematics is a solitary activity, done by individuals in isolation.
  - ❑ Students who understand mathematics will be able to solve any assigned problem in less than five minutes.
  - ❑ Mathematics have little to do with the real world.

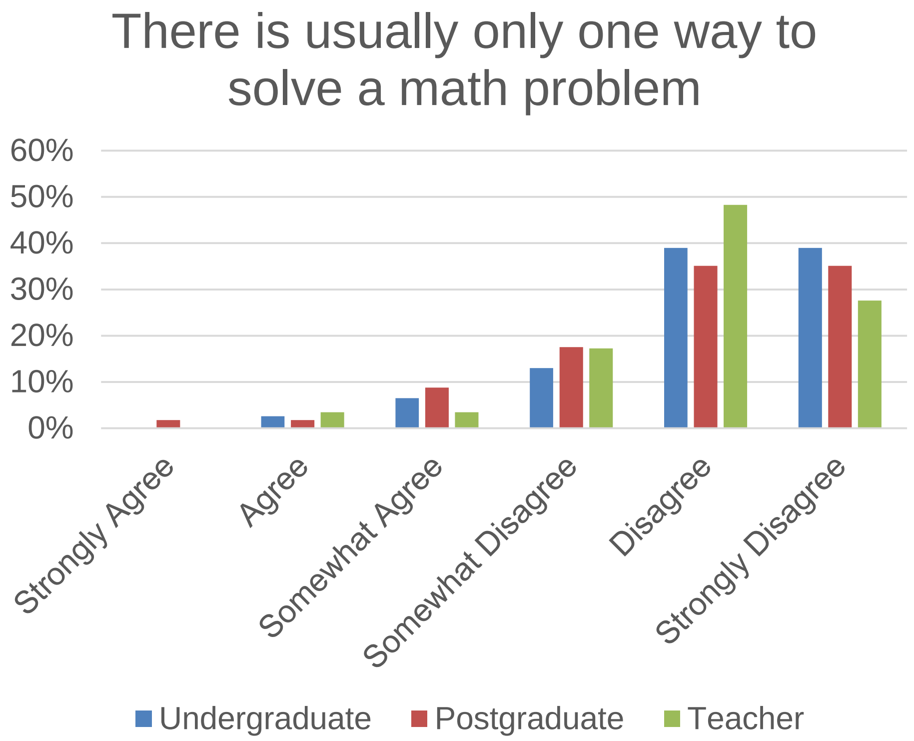
Mathematics Counselors

Myth #5: Misconceptions (I)



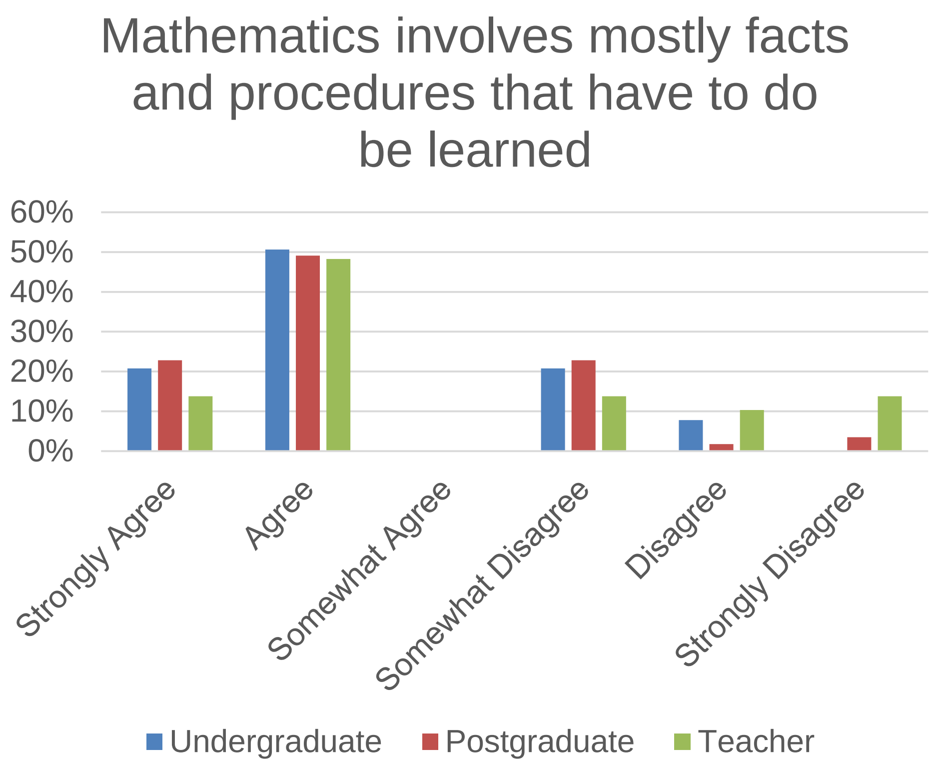
	Undergraduate	Postgraduate	Teachers
Agree	79%	79%	69%
Disagree	21%	21%	31%

Myth #5: Misconceptions (II)



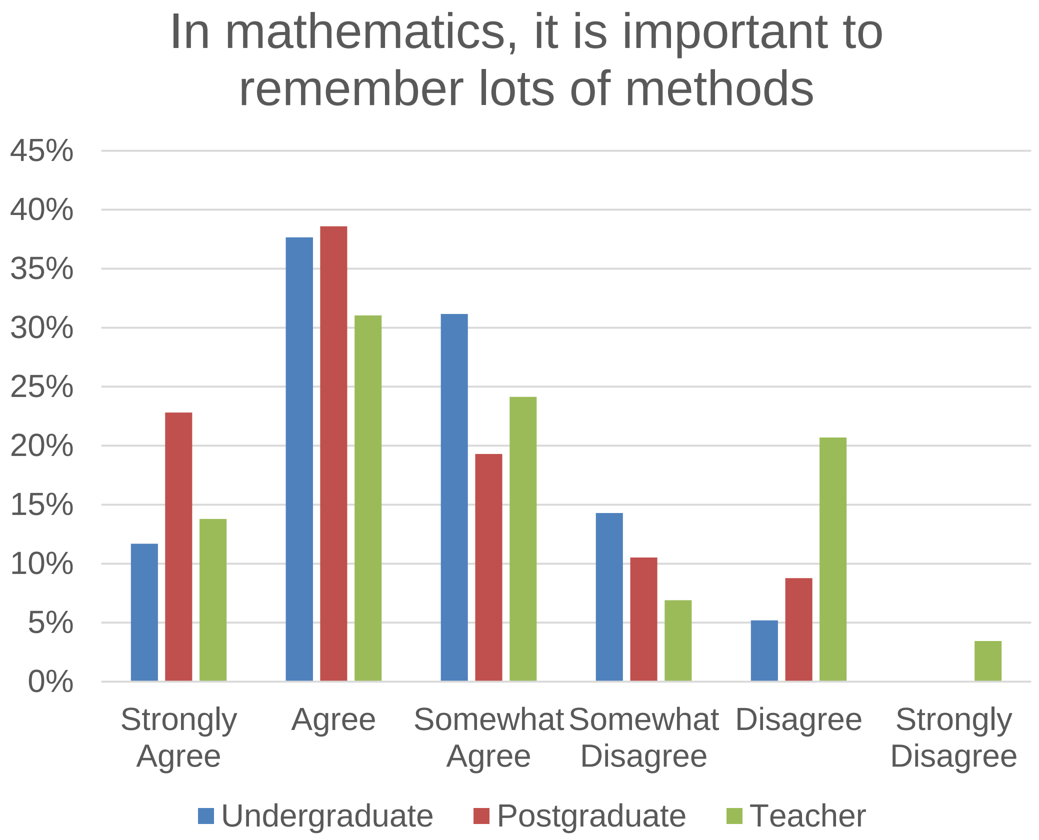
	Undergraduate	**Postgraduate**	**Teachers**
Agree	9%	12%	7%
Disagree	91%	88%	93%

Myth #5: Misconceptions (III)



	Undergraduat e	**Postgraduate**	**Teacher s**
Agree	71%	72%	62%
Disagree	29%	28%	38%

Myth #5: Misconceptions (IV)



	Undergraduate	Postgraduate	Teachers
Agree	81%	81%	69%
Disagree	19%	19%	31%

The Outline

- The Myths about Mathematics
- The planned actions within the framework of the iTEM project


It Takes Two
to Tango!

Future Actions

- **Personalized Education:** Address each student misconceptions and facilitate her math learning.
- **Engage students in their learning:** give them the opportunity to apply the power of knowledge.
- **Three stages of learning:** Romance, Precision and Generalization / Mathematics education at he moment fails to introduce the 'Romance'.
- Build curricula that involves **structurally rich activities (real life experiments) /** learn mathematics through experiments in electronics for example (Play with Mathematics)
- Incorporate '**Low floor, high ceiling tasks**'. – Introduce PBL and Scrum in Higher Education
- Math intelligence can be improved with hard work. Highlight the **effort and strategies to success** to your students!! Efficient guidance and devotion are the key of success.

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Future Actions – Suggestions from the Survey Participants

- Provide examples from the easiest to the harder ones (See Pythagoras – AI role in HE).
- Control the pace of teaching delivery / Address personal needs.
- Introduce creativity & collaboration along math teaching & learning.
- Link Math's with the discipline is taught.
- Promote the discipline and not just solved problems: The History of Math's, the link with real life problems. Show the process towards the proposed solution.

Future Actions – Suggestions from the Survey Participants

- Engage teachers with passion to teach, give time to their students to learn and be challenged.
- Boost the confidence of students' regarding their ability to learn and use maths.
- Promote the collaboration between students during studying Maths.
- Feel the gap between high and University knowledge math level.
- Introduce visualizations during teaching.

Top Five Soft Skills for the STEM Employers

care

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How to Learn Maths – for students

Help you to love and learn math to the highest levels
Mathematics mindset can be flourished everywhere

Associate Professor Konstantinos Petridis

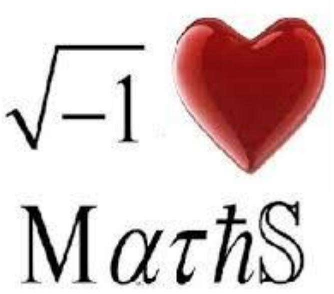
Department of Electronic Engineering, Hellenic Mediterranean University

How to Learn Maths

How to Teach Maths

The Outline

- *Knocking Down the Myths About Math*



Knocking Down the Myths About Maths

People they do not like Math, mainly because of the way it is taught

Knocking Down the Myths About Math

- Math in the majority of the cases offered as **a dry and monotonous subject;** instead it should be taught as **a breathing subject with a lot of ways to access it**

- **Myths about Maths:**

1. Only some people can be good in maths
2. Maths are not linked with real life applications
3. Maths are boring

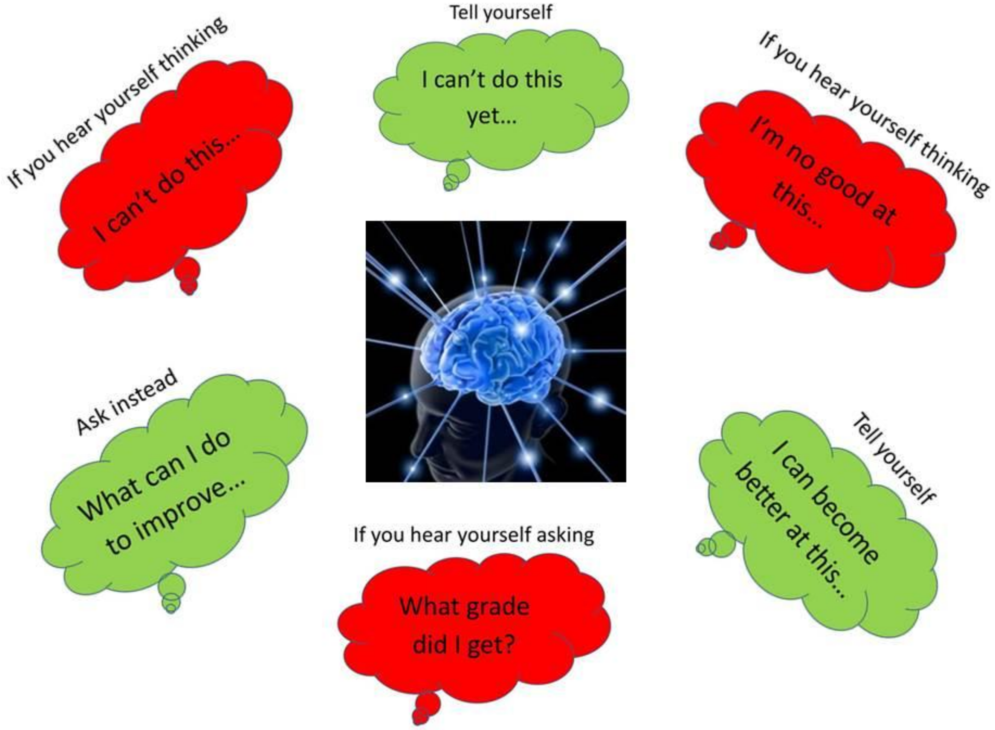
Knocking Downs the Myths About Maths

Knocking Down the Myths About Maths

- **But the truth is the following**:

1. The maths are everywhere & and are linked with any subject
2. Anyone can learn, understand and perform in Mathematics – learning depends on your experiences. By practicing the brain is getting larger
3. The problem is the way Maths are taught in the University!!!!

How to Learn Maths



Math and Mindset

Math & Mindset

- **Mindset:** your beliefs about you have an impact on math learning

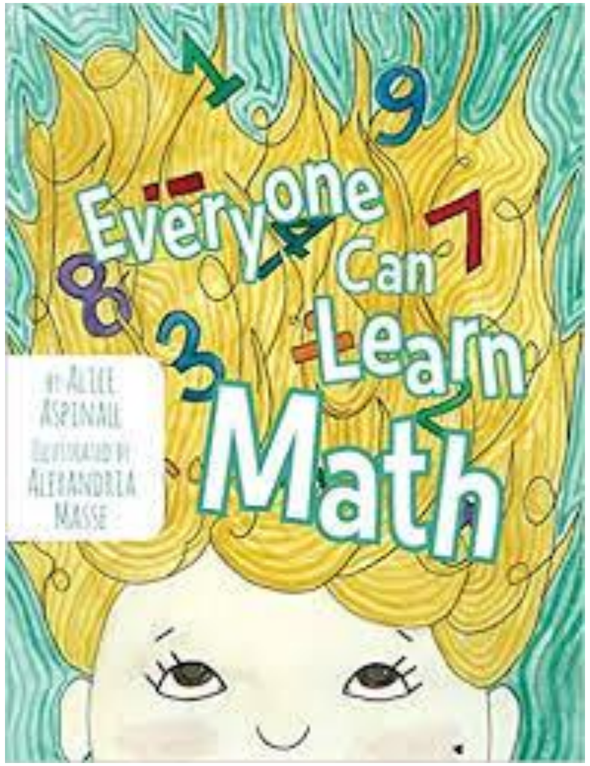
- There are **two types of mindsets** (according to Carol Dweck):

1. Growth mindset: your smartness increases with hard work
2. Fixed mindset: you believe that you are either smart or you are not or you can do better with hard work but you cannot change your basic level of intelligence

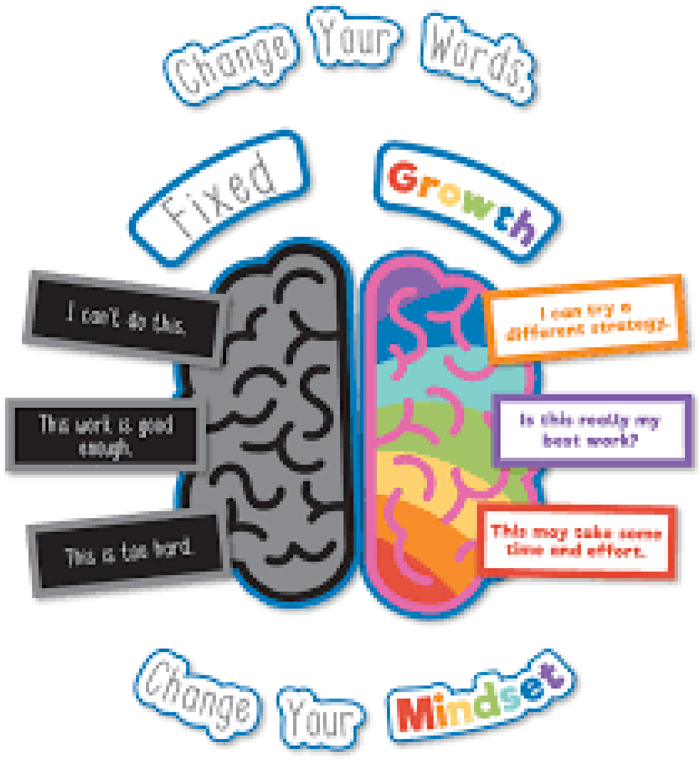
- **Classroom Question**: What do you think? What do you think about learning or using Mathematics?

Math & Mindset

- It is very important to have a **growth mindset to learn Maths;** in order do not give up in the 1st difficulty



Math & Mindset



- The people with growth mindset are doing better in maths because:

1. They try harder and longer
2. They see their failures as an opportunity to learn more things
3. They persist and they do not give it up when something is hard

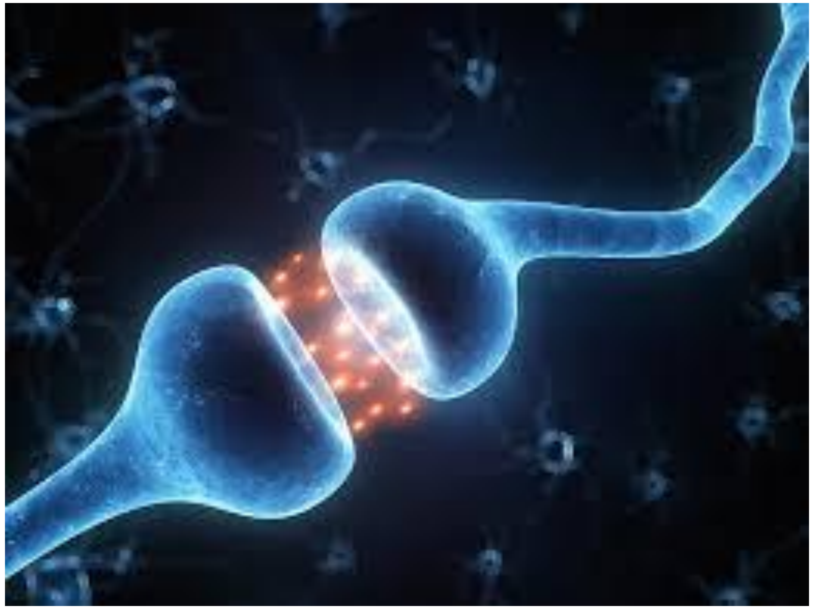
Math & Mindset

So **a message about Math and you** is the following:

Math & Mindset

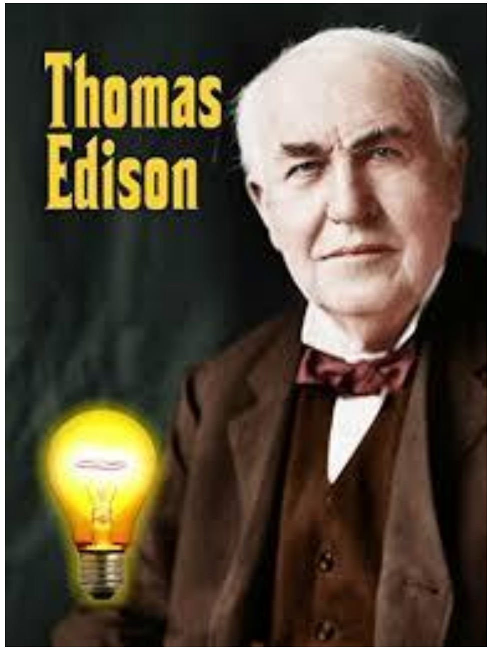
Classroom question: Why do you think that when students were praised for being smart they then chose the easy problem?

Mistakes – New Evidence



- **When you make a mistake in maths your brain grows**; since you do not need to get the right answer, you just need to think of the mistake for synapses to fire!
- It is important **to get involved in hard problems** that encourage you to struggle and even make mistakes as there are the most important times for your brain

Mistakes New Evidence



Question: *'How do you feel after 1000 times of failing before the invention of the electrical bulb?* ***Edison:*** *I was not failing but I was taking steps ahead*

'Genius is one percent inspiration, and ninety nine percent perspiration'

Mistakes New Evidence

Math and Speed

- What is the relation between Math and F1?
- **None or almost irrelevant**
- **‘Being good in Math does not mean being fast in maths. In fact the opposite is true’**
- Being good in maths means to be able to relate things to each other – to compile thinks!

Number Flexibility

Wrong Ideas About Maths

- Maths involve just methods & rules
- You should be fast with Maths
- Maths is a subject that can be taught having a book & a teacher

Number Flexibility

- A secret to learn maths is **your flexibility with numbers**
- **High Achievers are more flexible with numbers than low achievers**
- Mistakes makes our brains to grow

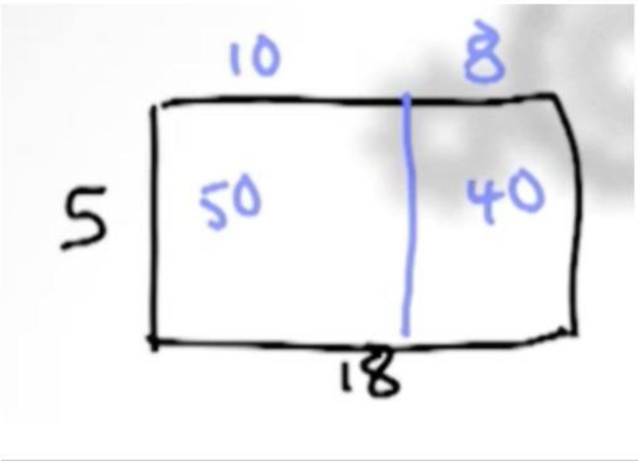
Number Flexibility

- Use your creativity to reach an answer
- **There are more than one way to solve a problem**
- When a lecturer teaches you how to solve a problem try to discover how to reach the same result following different path

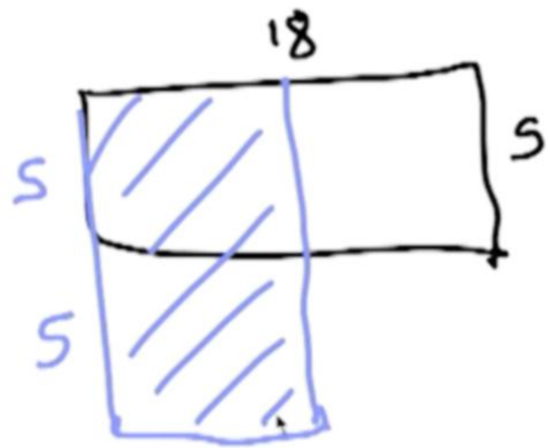
Number Flexibility

How much does it count 18*5 – Different Options / Use the most friendly numbers

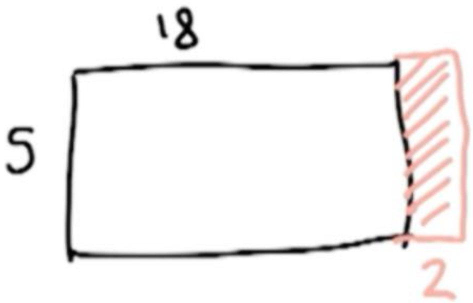
Option A



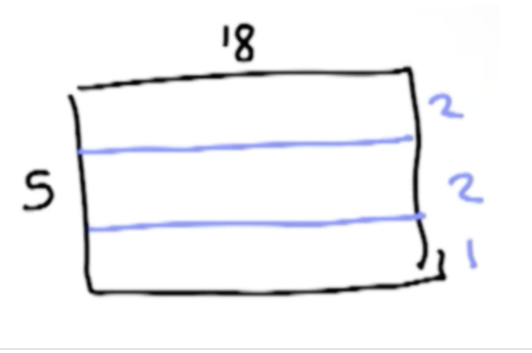
Option B



Option C



Option D



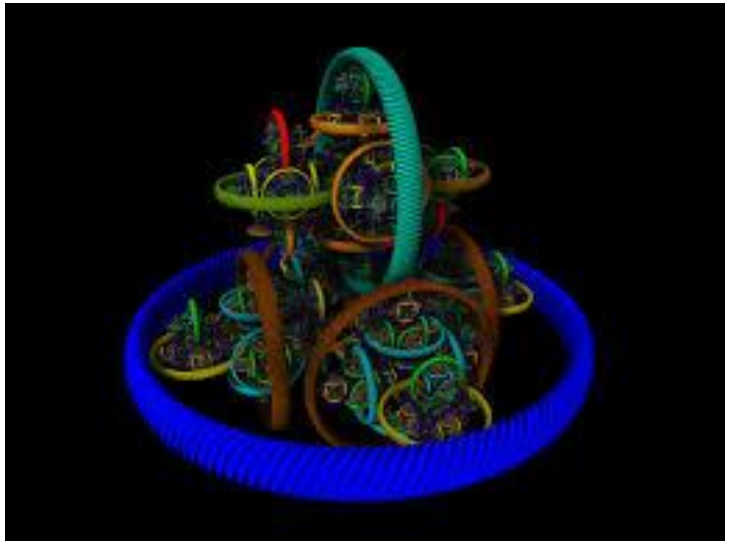
Reasoning

- **Talk about Math**; this will make the difference
- **Try to discuss and study Maths** with others and not by yourself
- Students **love to learn Maths by each other** for many reasons

Reasoning

- Discussing about Maths is very crucial since at the same time you are **reasoning**
- **Reasoning:** *talk through your methods and give reasons for your choices; Reasoning* ***allows people to connect ideas*** *and make mathematical breakthroughs*
- **Math Mindset** is one of the most crucial assets for your employability along the digital era we live

Mathematical Connection



- **Mathematical connection** together with flexibility and reasoning is the 3rd ingredient of success along learning mathematics
- **For example** think in how many different ways we can describe a function: formula, graph, table and link in a daily life example
- **PISA results** shows that the high achievers link mathematics in real world whereas the low achievers try to memorize formulas and techniques

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Chapter One: Functions

An Introduction – A World in Notation

*Mathematics is not just a set of tools with which you can calculate things, **it is also a language** in which you can make very precise statements about, for example, numbers. This language uses its own symbols and while we do not think you should be able to use them yourselves yet, you might encounter them a few times in this course and perhaps also in more advanced courses you take later. You can use this entry both as an introduction to the symbols, and as as a reference.*

An Introduction – A World in Notation

Depending on the context you would like to talk about all fractions, or just about integers. As such it is convenient to have a nice name and symbol for a few of the most common sets

- **N** is the set of Natural numbers, that are all positive numbers. **Examples** are 1, 13, and 23852
- **Z** is the set of the integer numbers. **Examples** are -5, -53, and 78
- **Q** is the set of all rational numbers. **Examples** are −73 , 234159023 , and 9
- **R** is the set of all Real numbers. **Examples** are e^2, π
- **C** is the set of complex numbers. **Examples** are 3+j2

An Introduction – A World in Notation

*In calculus many of the sets we consider are **intervals:** sets which contain all real numbers in between their starting point and their ending point. An interval can either include or exclude its starting and ending points, and several different notations for this are in use in different sources*

- **(a,b)** is the set of all x with $a < x < b$ (thus excluding the boundary points)
- **[a,b]** is the set of all x with $a \le x \le b$ (thus including the boundary points)
- **(a,b]** is the set all x with $a < x \le b$ (thus only including b, but excluding a). Likewise we have [a,b)
- **[a,∞)** is the set of all x with $a \le x$. The symbol ∞ denotes infinity: a quantity larger than any real number. Note that infinity is itself not a number, so it can not be included in an interval. Similarly we have (a,∞), $(-\infty,b]$ and $(-\infty,b)$
- **(−∞,∞)** is the set of all x (i.e. R)

An Introduction – A World in Notation

Element of: ∈

One thing you want to say over and over again is a sentence like " x is a real number", or " x is in the interval $[-2,4)$ " (that is: " $-2 \le x < 4$ "). To abbreviate this statement we use the "element of"-symbol $\in$. Thus the above two sentences can be written as " $x \in R$," respectively " $x \in [-2,4)$."

We can also say " x is not an integer" or " x is not in the interval $(-\infty,0)$." by striking through the "element of"-symbol as follows: " $x \notin Z$ ", respectively " $x \notin (-\infty,0)$

Chapter One: **Functions**

The Outline

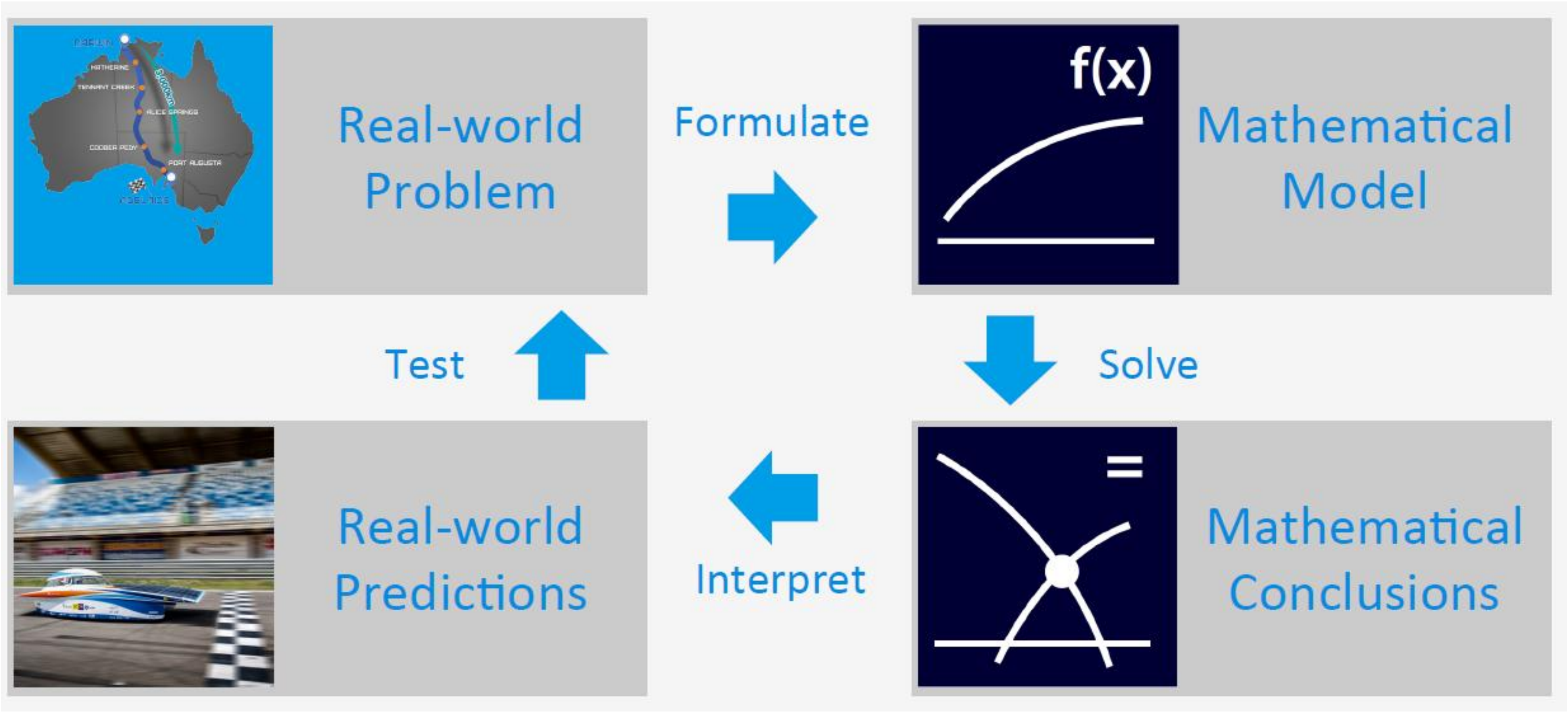
- The role of functions in electronics
- The definition of functions
- The properties of function: domain, range and continuity

It is essential to thoroughly understand mathematics in order to become an engineer !!!

Functions

- To face a **real life challenge** you need to break down it into smaller parts: the **parameters/factors** that determine/govern the problem
- These factors are linked together **through functions: polynomial, exponential and trigonometric functions**
- After the modelling the challenge, you need to solve the functions and reach conclusions, test the conclusion under real life conditions

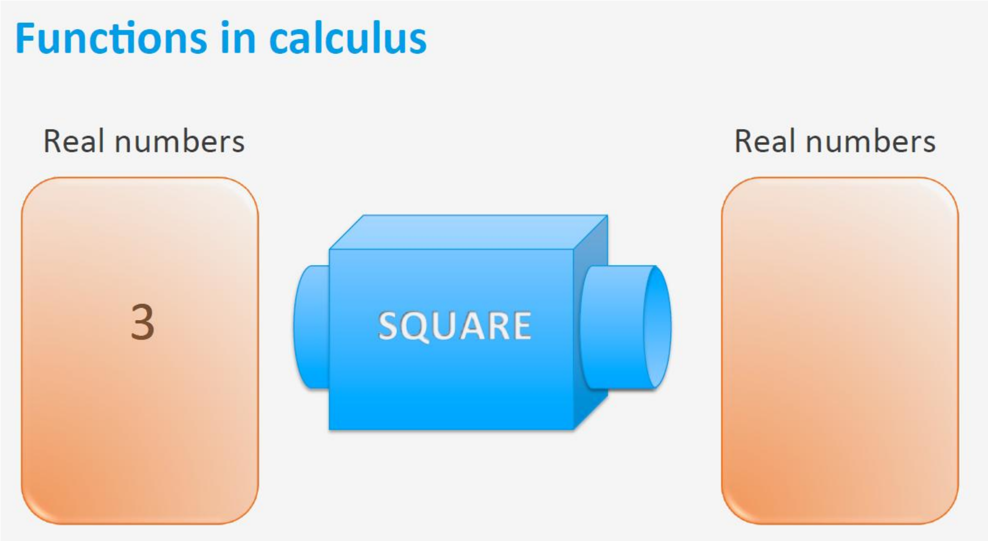
Functions



The above mathematical modelling cycle can be applied in any real – life context

Functions

- **What is a function?**



A function is a 'machine' that takes values from domain and generates a single value (belongs to the codomain of the function

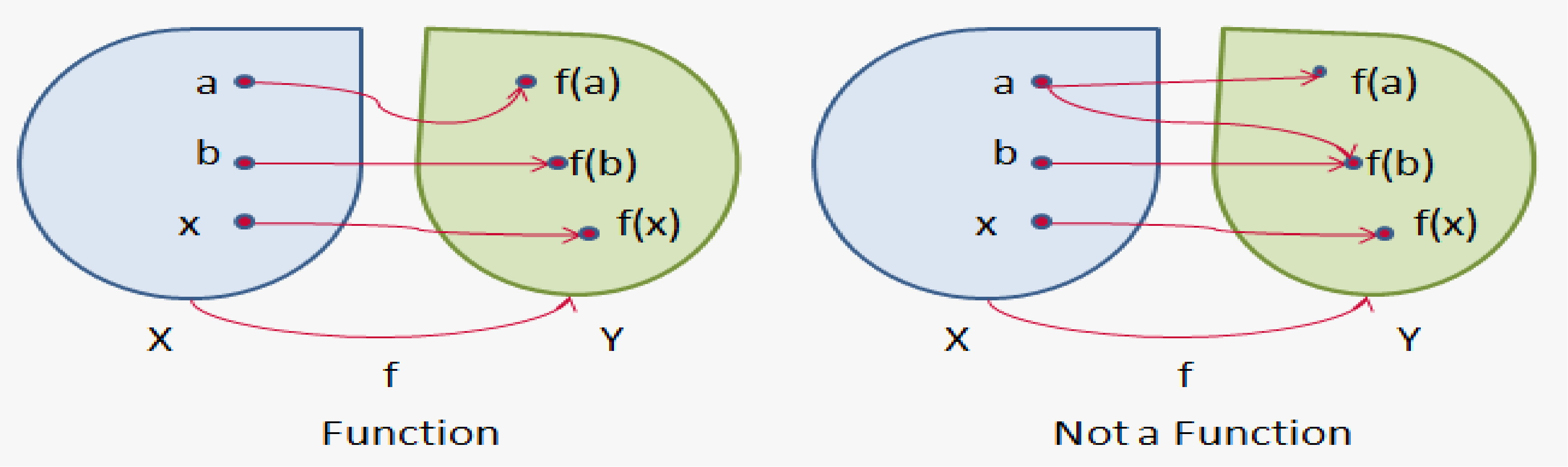
Functions

- **What is a function?**

Examples of functions in electronics:

- *The diode*
- *The operational amplifier*
- *The active filters*
- *The transfer function of any amplifier – the Bode Diagrams*

Functions



Independent – dependent variables

$$y = f(x)$$

$f(0) = 3$

$f(1) = 4$

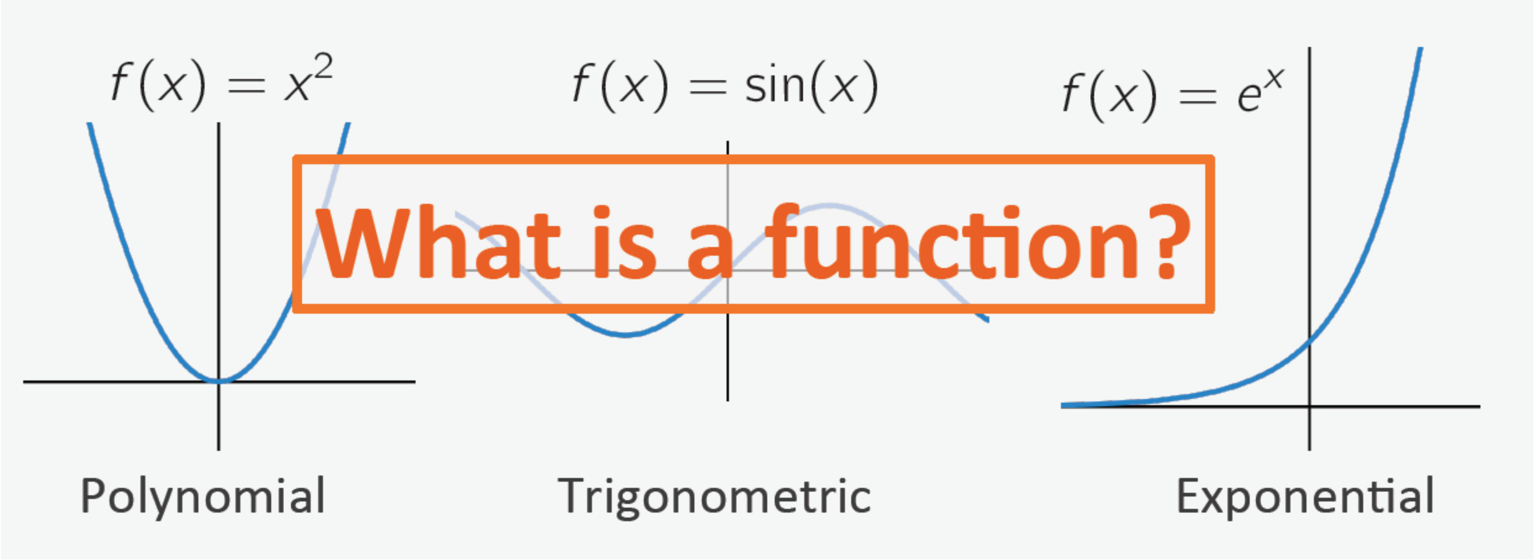
$f(2) = -1$

$f(3) = 6$

x	0	1	2	3
y	3	4	-1	6

Functions

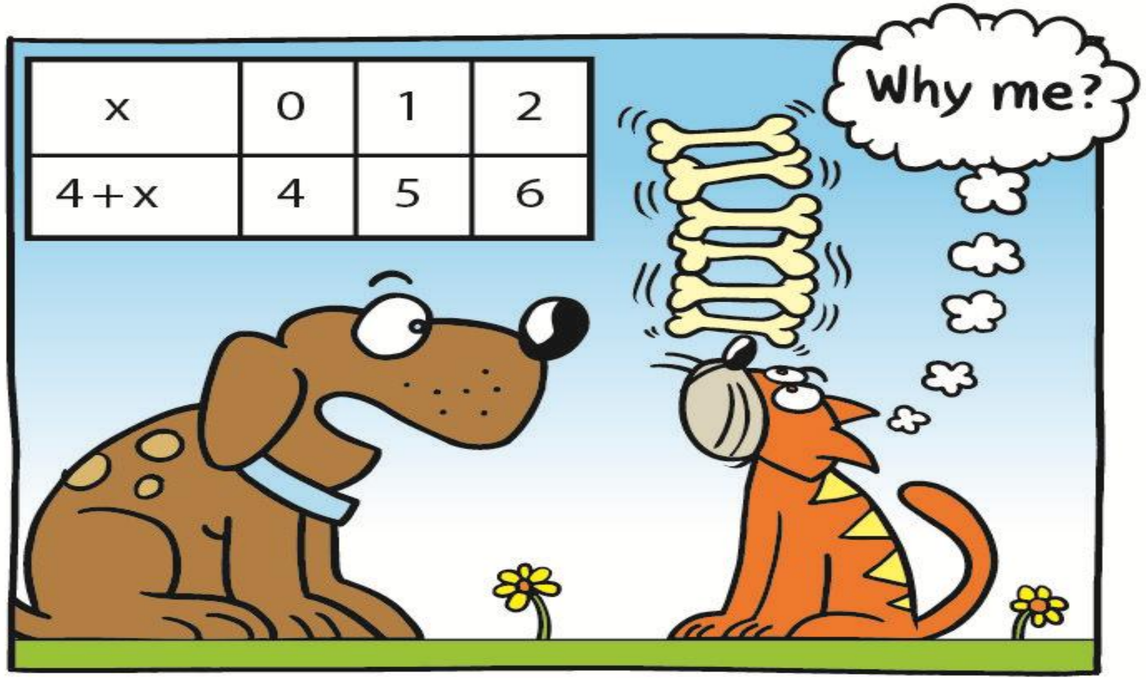
- **Types of functions:**



Exponential function in electronics: *the charging of a capacitor*

Trigonometric functions in electronics: *the electric field of an e/m wave*

Polynomial function in electronics: *??????????*



"Great! You're up to x = 2.
Let's keep going."

Functions

- **How to describe a function?**

a) Through **a formula** e.g. $f(x) = x^2$. The disadvantage in this case is that real – life situations is not easy to be associated with a formula

b) To counter-attack the aforementioned drawback, we describe a function using **a table** (input/output columns). The drawback in this case is the limited number of inputs/outputs data. So we have an incomplete description of the function

c) A third way is through **a graph**. With a graph you can identify also the properties of the function e.g. where the maximum or the minimum appears, how fast the function increases or decreases. The drawback is that you can not find exact values

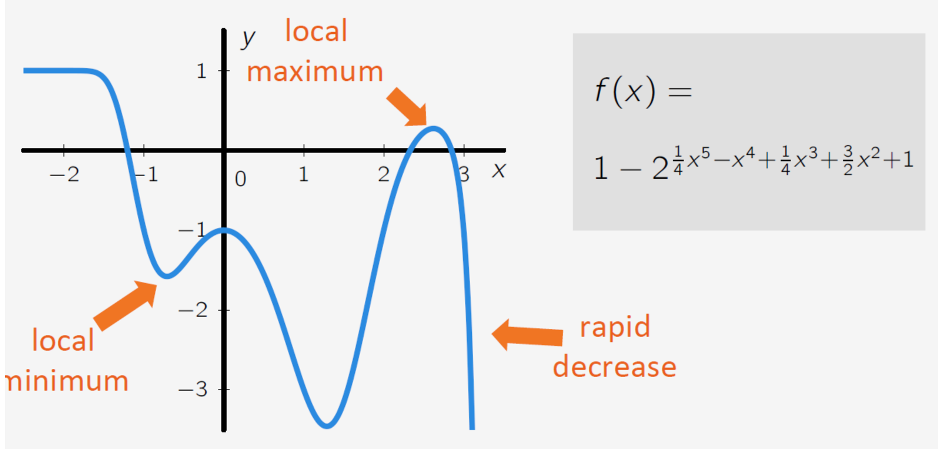
d) Finally we can describe a function **using wording**

Functions

The function

$$f(x) = x^2$$

Year	World population (mln)
1990	5,263
1995	5,674
2000	6,070
2005	6,454
2010	6,972



The Formula **A Table** **A graph**

In words: floor(x) is the largest integer $\leq$ x

Functions

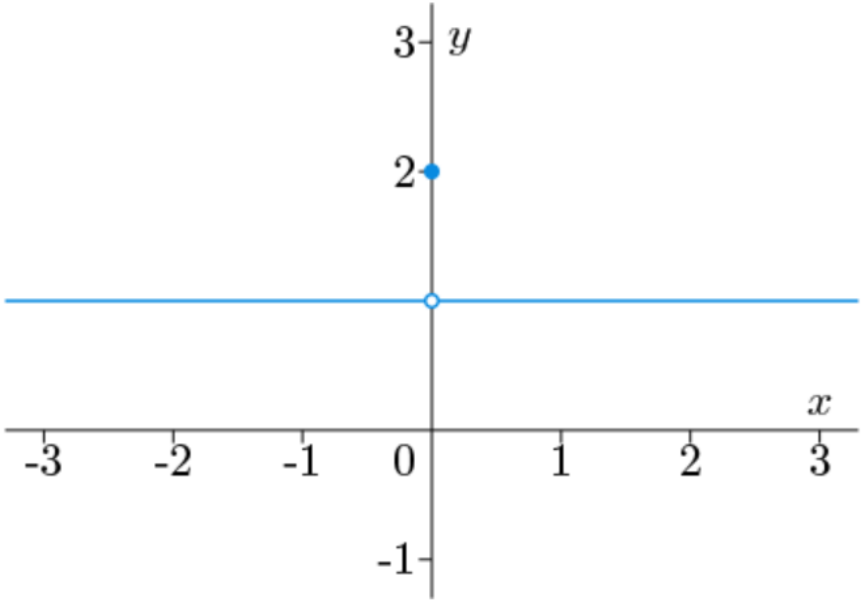
- **Note One**: Sometimes, it is more convenient to describe **a function by several formulas instead of one**. For example, consider the absolute value function abs(x) . If you want to describe what it does, it is convenient to distinguish between $x < 0$ and $x \geq 0$:

$$\text{abs}(x) = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

However we have one function we need two formulas to fully describe it!!

Functions

- **Note Two**: When drawing a graph, you sometimes want to indicate explicitly that a certain point is not part of the graph. A common way to do this is to use an open circle. For example, consider the function f that is 1 for all non-zero x, and 2 if x is 0. The graph woul



The open circle indicates that the line does not continue there, but jumps to y=2. This value is indicated by the blue dot.

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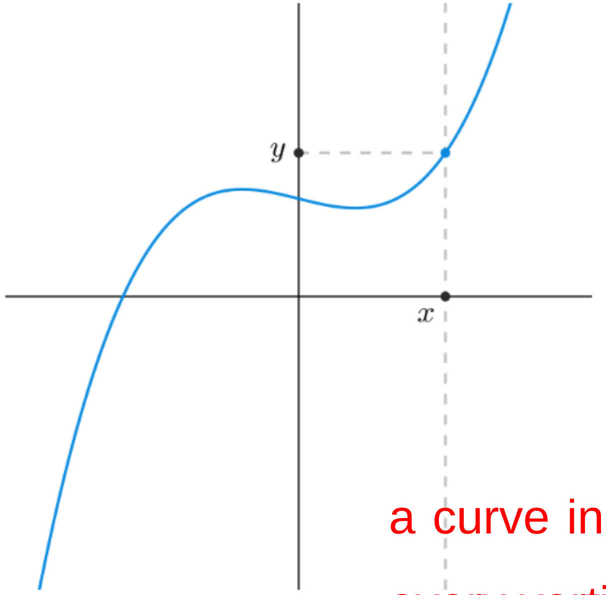
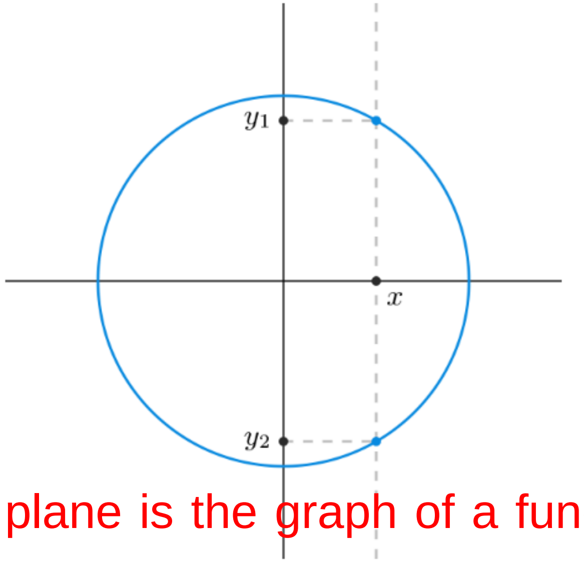
Pythagoras
2021-1-RO01-KA220-HED-000032258

Functions

- **Note Three**: Not every curve in the plane is the graph of a function. We have seen that for a function, there should be only one output per input. So for a given x , there can be at most one y such that the point (x,y) lies on the graph of the function.

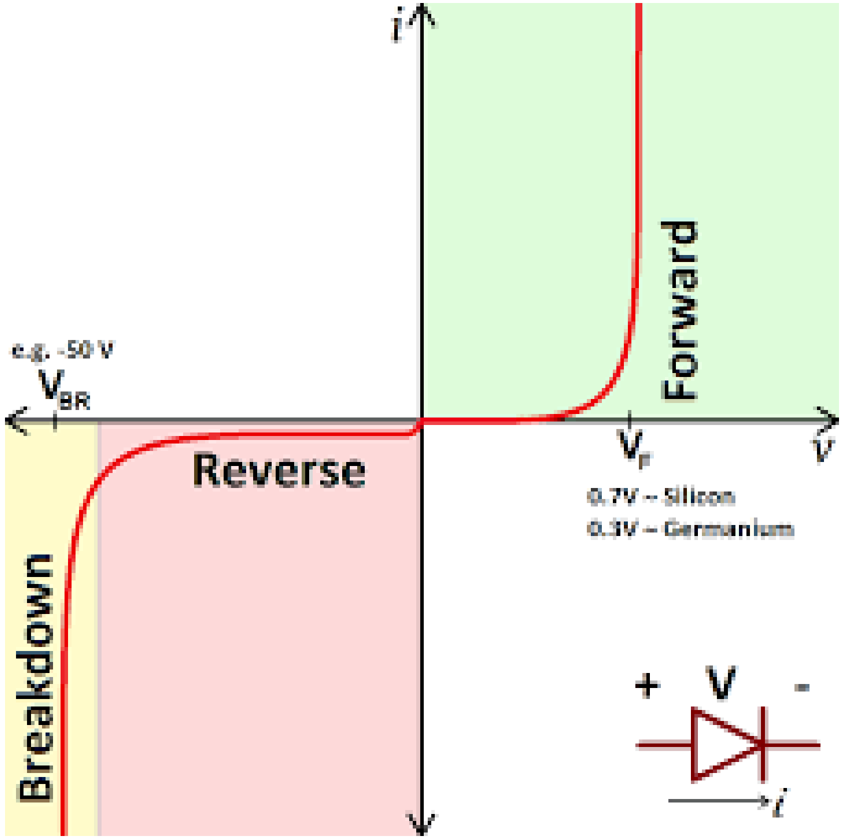




But, for example, for a circle this is not true. Indeed, for some values of x , there are two y -values such that (x,y) lies on the circle. **So a circle cannot be the graph of one function**

a curve in the plane is the graph of a function precisely if every vertical line intersects the figure in at most one point

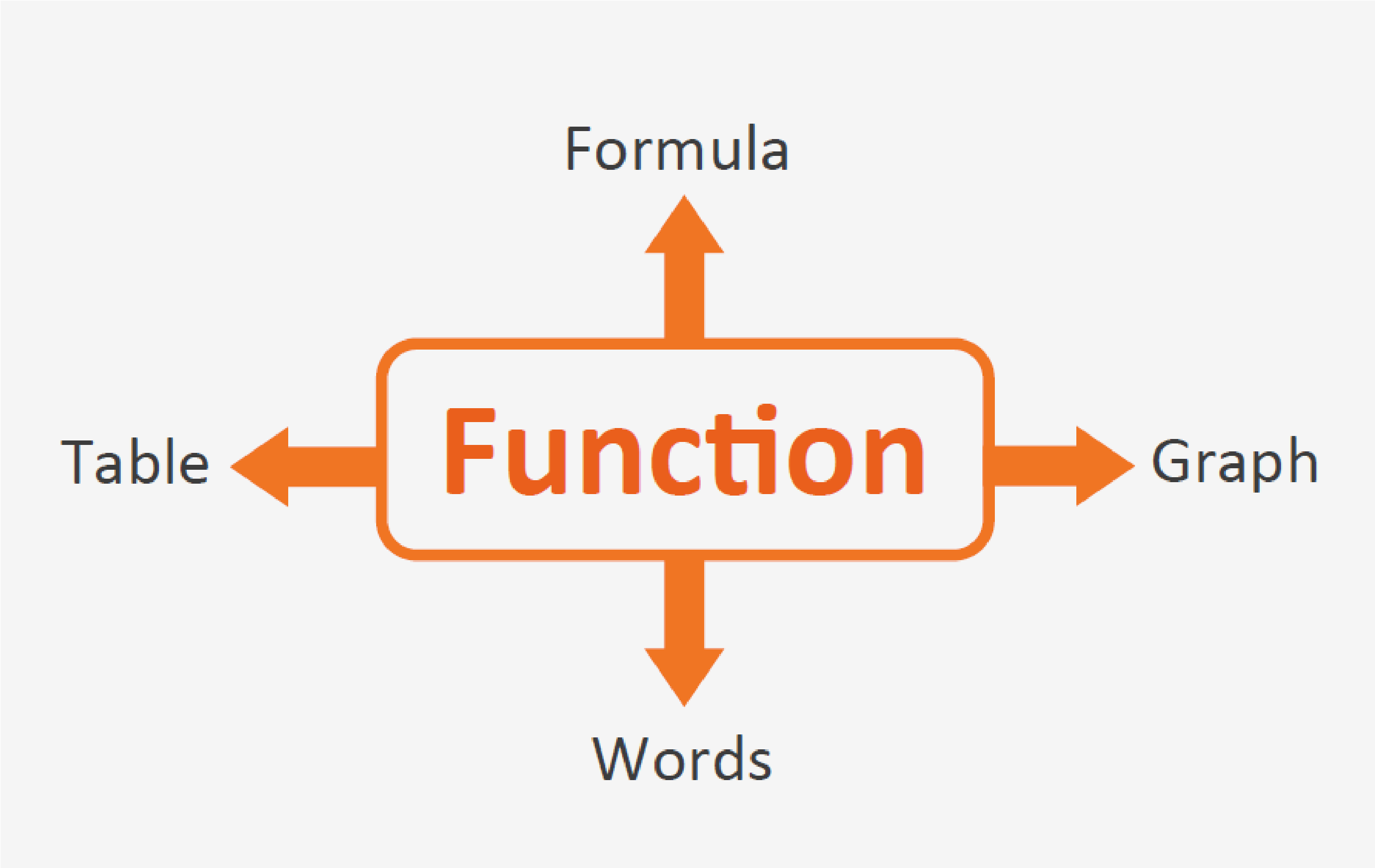
Functions



$$I_D = I_S \cdot \left(e^{\frac{V_D}{\eta kT / e}} - 1 \right)$$

	Experimental Data		Single Diode Model CIABC		Double Diode Model CIABC	
Data	$V_{tm}(V)$	$I_{tm}(A)$	$I_{te}(A)$	R_{err}	$I_{te}(A)$	R_{err}
1	−0.2057	0.7640	0.7641	−0.0001	0.7640	0.0000
2	−0.1291	0.7620	0.7627	−0.0007	0.7626	−0.0006
3	−0.0588	0.7605	0.7614	−0.0009	0.7613	−0.0008
4	0.0057	0.7605	0.7602	0.0003	0.7602	0.0003
5	0.0646	0.7600	0.7591	0.0009	0.7591	0.0009
6	0.1185	0.7590	0.7580	0.0010	0.7581	0.0009
7	0.1678	0.7570	0.7571	−0.0001	0.7572	−0.0002
8	0.2132	0.7570	0.7561	0.0009	0.7564	0.0006
9	0.2545	0.7555	0.7551	0.0004	0.7555	−0.0000
10	0.2924	0.7540	0.7537	0.0003	0.7547	−0.0007
11	0.3269	0.7505	0.7514	−0.0009	0.7537	−0.0032
12	0.3585	0.7465	0.7474	−0.0009	0.7526	−0.0061

Functions



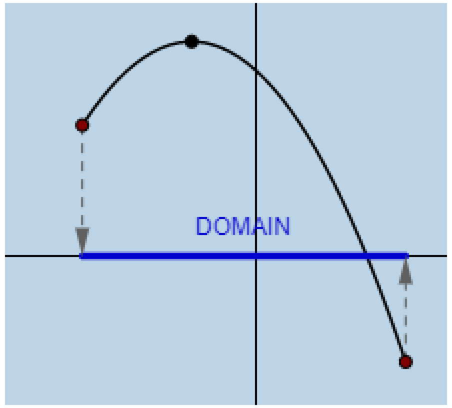
Domain, codomain and range

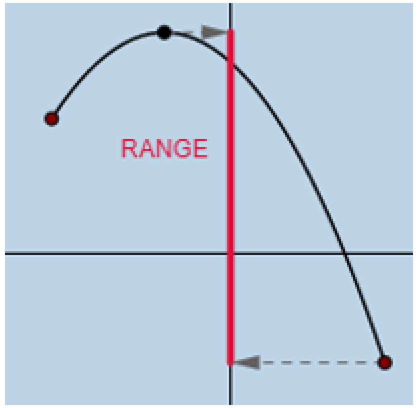
- **What is the domain, codomain and range of a function**? **What numbers can you put in a function and what is the influence of the context**?

- To be able to understand the properties of a function we should have an idea **of the input values** for which the function can be defined – the domain of the function

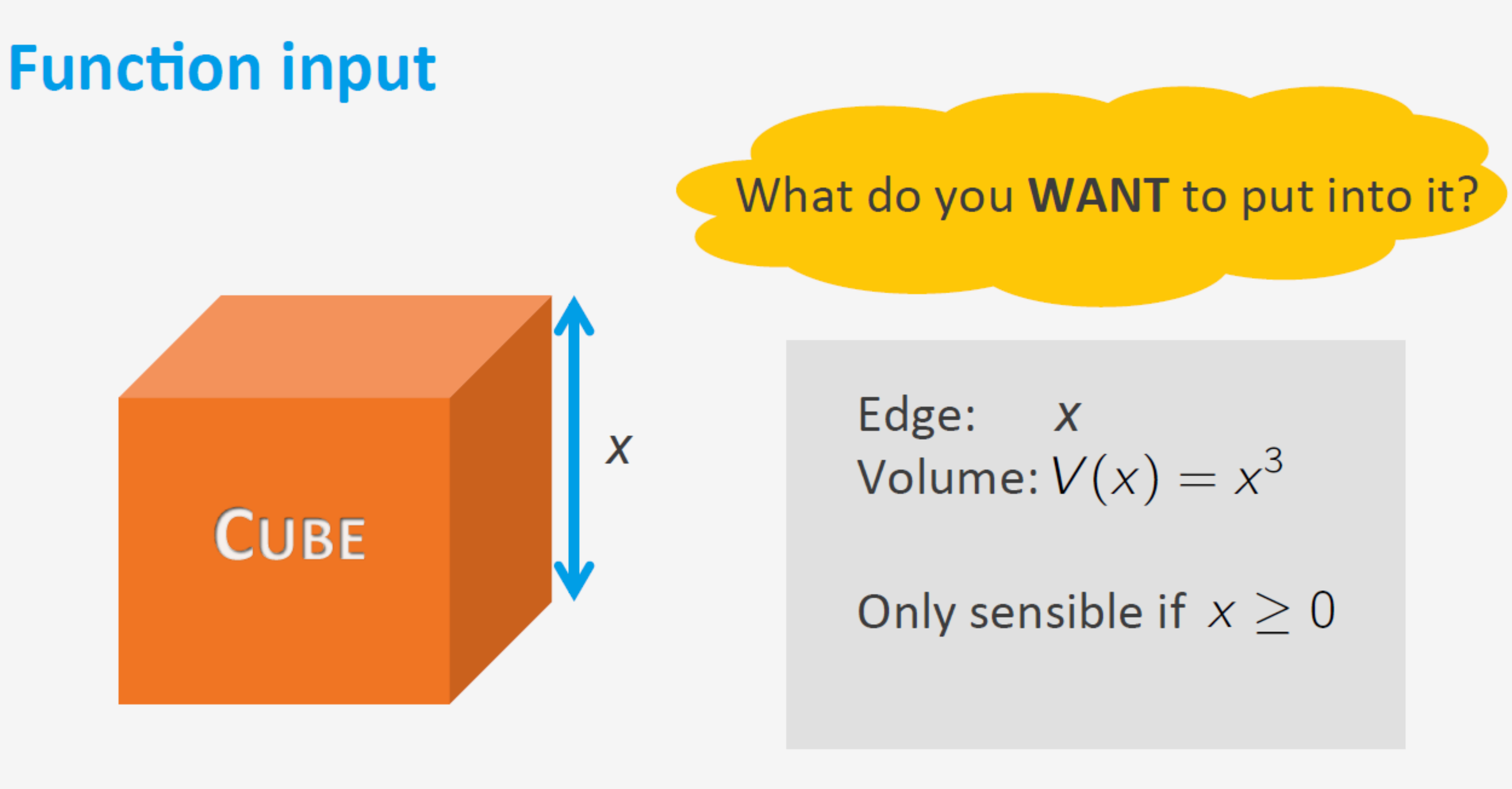
- **The maximum domain** is the range of values a function can take, considering any possible limitations

Domain, codomain and range



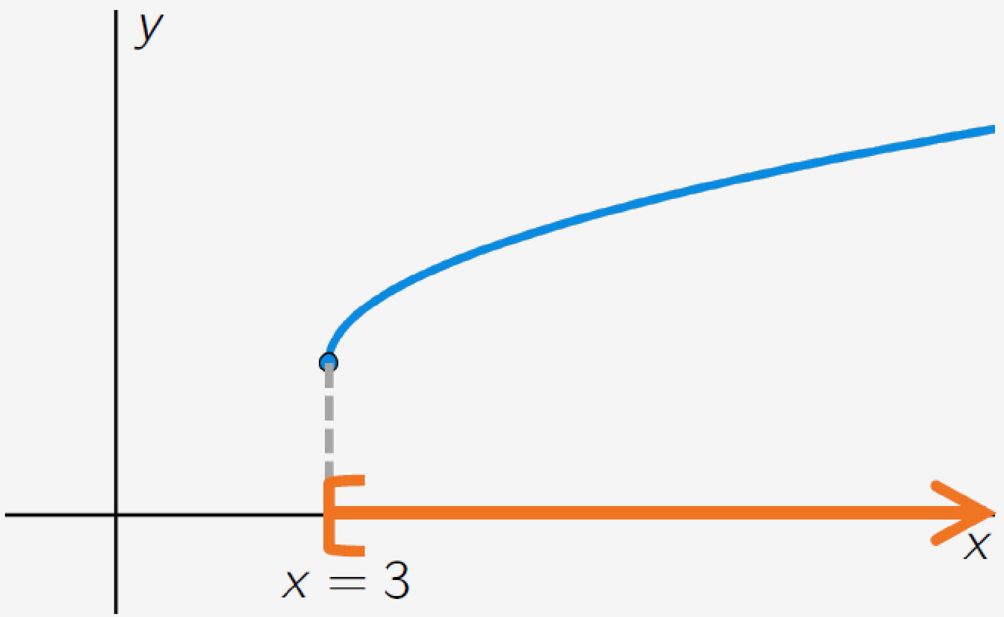


Domain, codomain and range



Domain, codomain and range

Maximal domain



$$f(x) = 2 + \sqrt{x - 3}$$

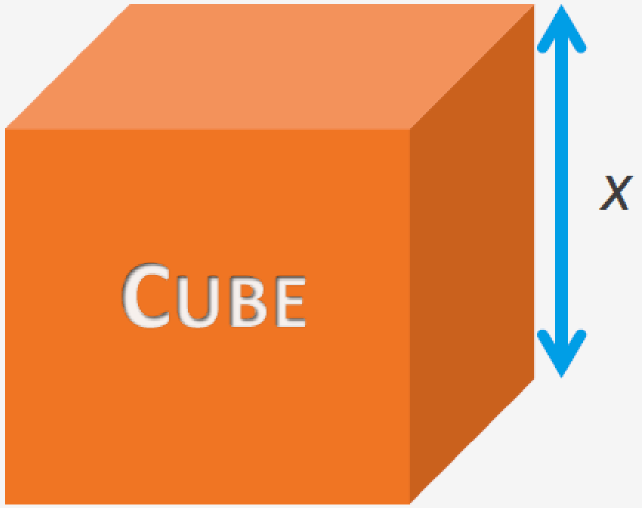
$$\geq 0$$

Only condition: $x \geq 3$

Maximal domain: $[3, \infty)$

Domain, codomain and range

Natural domain

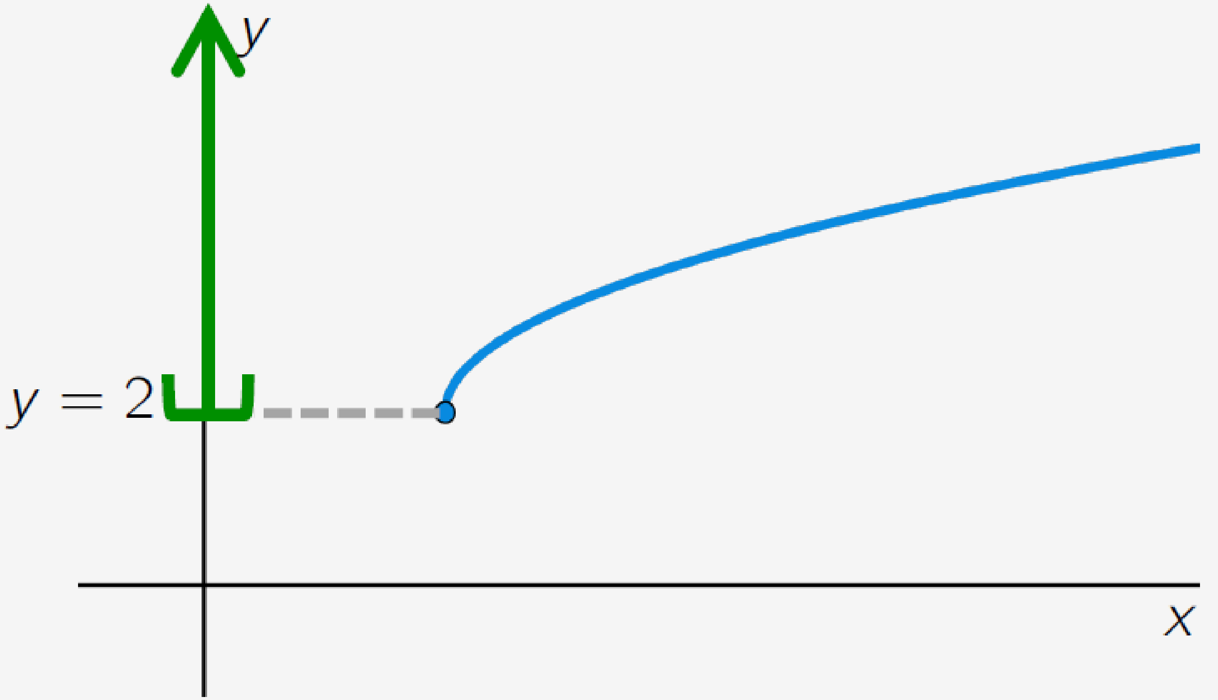


Edge: x
Volume: $V(x) = x^3$

Maximal domain: $\mathbb{R}$

Natural domain: $[0, \infty)$

Domain, codomain and range



$$f(x) = 2 + \sqrt{x-3}$$

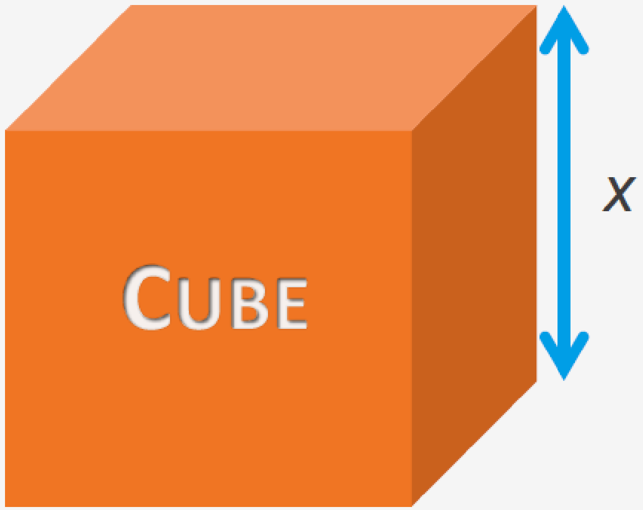
Range: $[2, \infty)$

In general:

For which y does $f(x) = y$ have a solution?

Domain, codomain and range

Range depends on domain



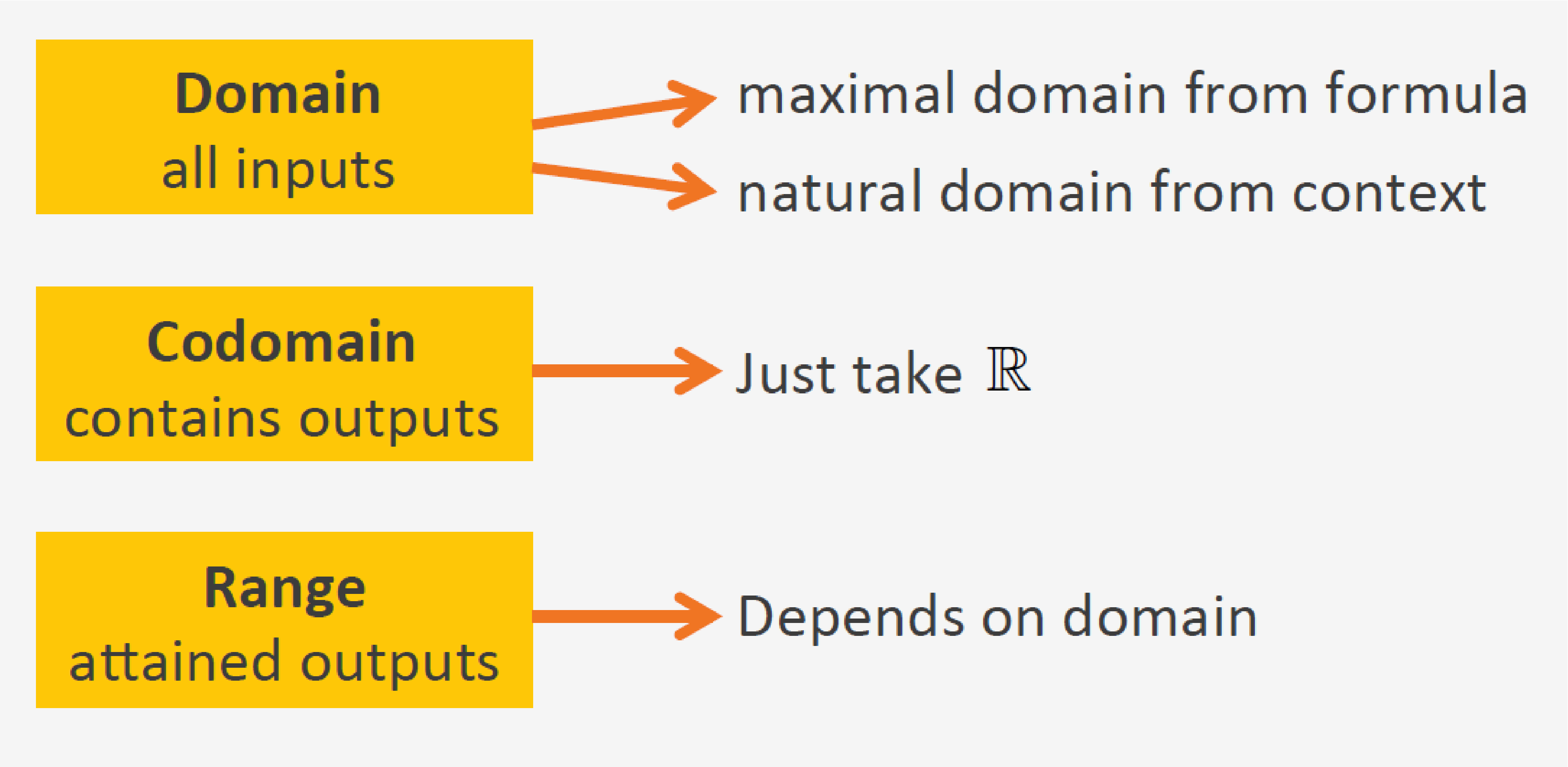
Edge: x

Volume: $V(x) = x^3$

Natural domain: $[0, \infty)$

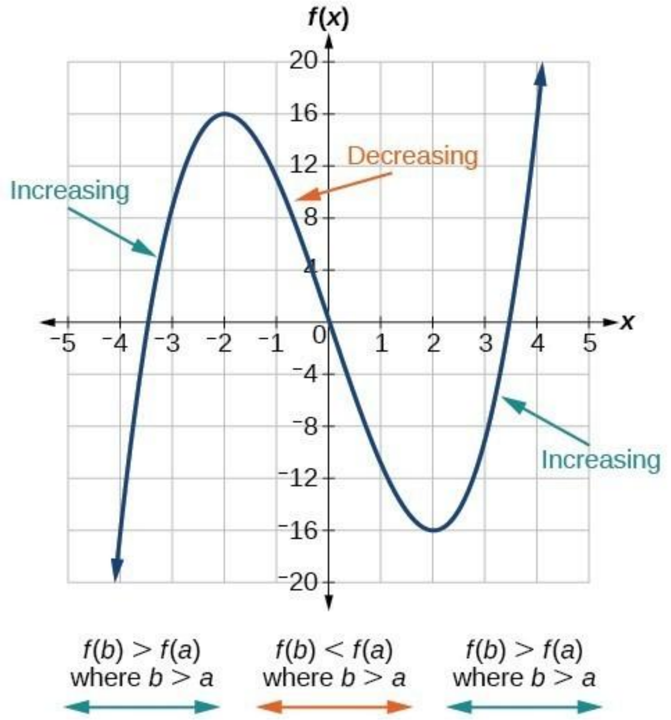
Range: $[0, \infty)$

Domain, codomain and range



Function Increase and Decrease

DEFINITION: If a function $y = f(x)$ is defined on an interval $[a, b]$, then $f(x)$ is *increasing on* $[a, b]$, if, whenever $x_1 < x_2$, then $f(x_1) < f(x_2)$. The function *f* (*x*) is *decreasing on* $[a, b]$, if, whenever $x_1 < x_2$, then $f(x_1) > f(x_2)$.

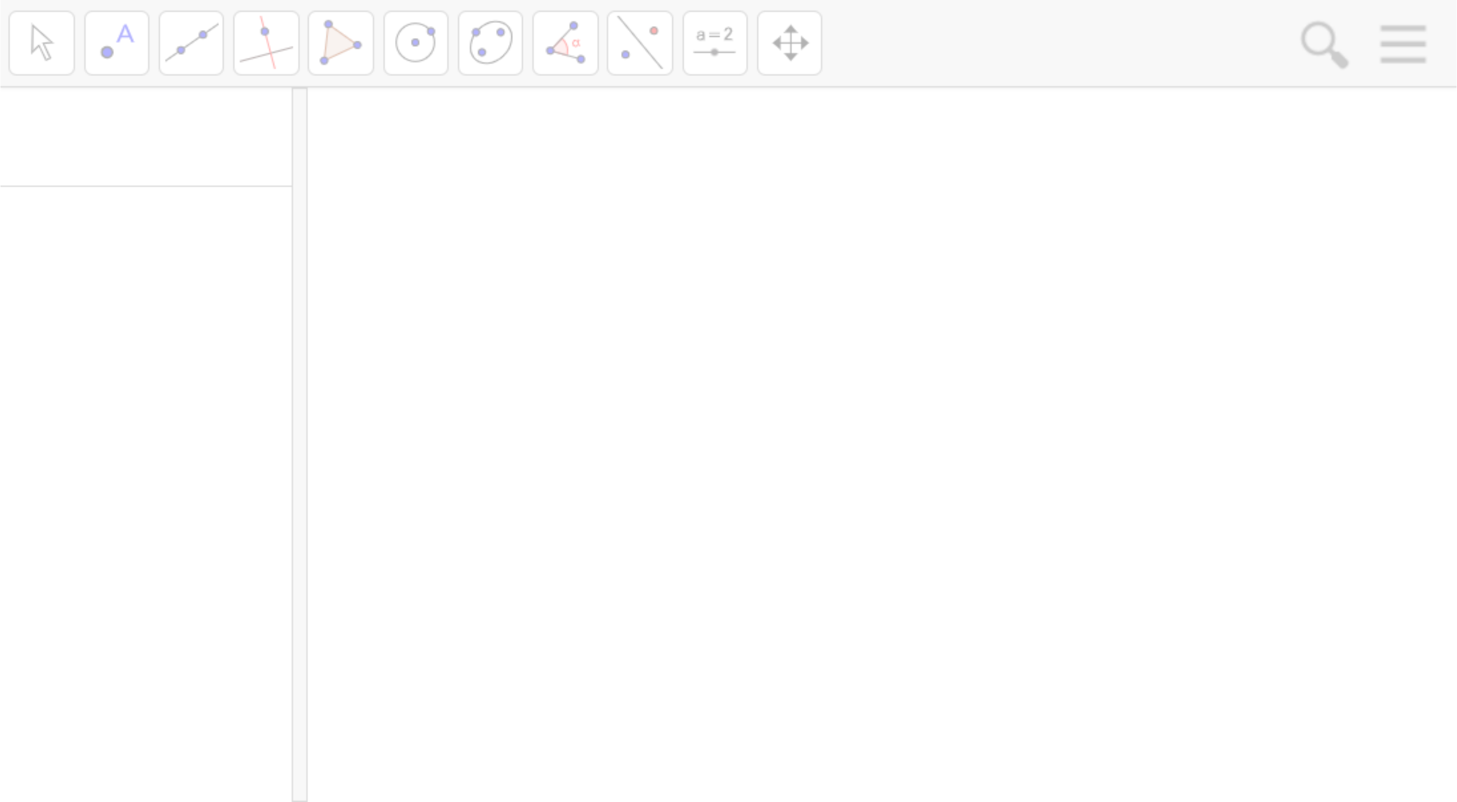


Function Increase and Decrease

Exploration 1.2.2: Function Increase and Decrease

Consider the graph of $y = (x-1)(x-2)(x-3)$ for the following problems.

- Is this graph increase in the domain [3,4]
- Is this graph increase in the domain [2,4]



Build your Function

4. On your own sheet of paper draw a function $y = f(x)$ with following properties:

- The domain is $(-\infty, 3]$
- The range is $[-1, \infty)$
- $f(0) = 1$
- The x-intercepts are at -1 and 1

Graphs

The Rectangular Coordinates

- We locate a point on a real number **line by assigning a number.**
- We locate a point in a two-dimensional plane, we locate points **by using two numbers**.
- The **rectangular or the Cartesian coordinate system**; the **coordinate axes**.
- The **coordinates are an ordered pair (x,y) of real numbers**.
- All the distances between two points can be measured using this system.

Figure 1

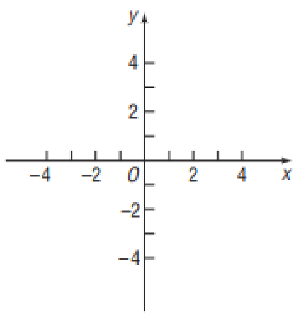


Figure 2

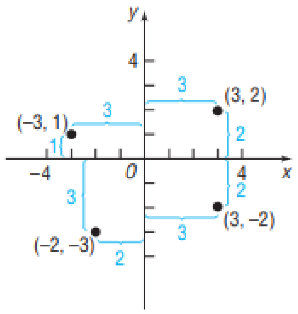
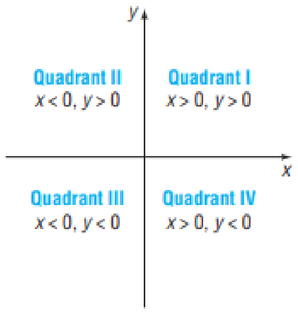


Figure 3



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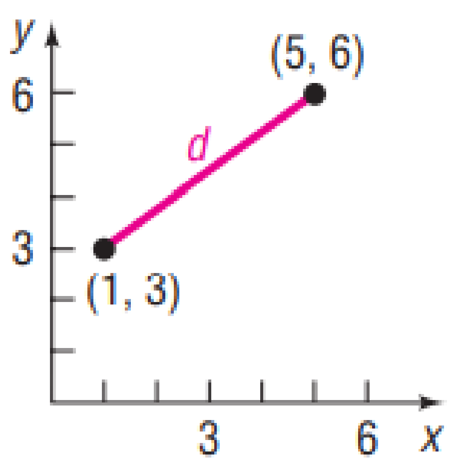
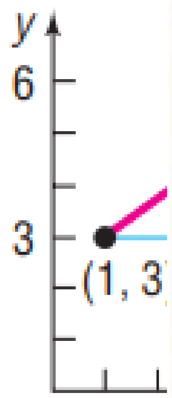
Examples

Example #1: Finding the distance between two points.

First, plot points (1,3) and (5,6) and connect them with a straight line. Use the Pythagorean Theorem to calculate the distance.





In Words
To compute the distance between two points, find the difference of the x-coordinates, square it, and add this to the square of the difference of the y-coordinates. The square root of this sum is the distance.

…n two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, denoted by

$$(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \textbf{(1)}$$

Examples

Example #2: Finding the distance between two points.

First, the distance d between points (-4,5) and (3,2).

Examples

Example #3: Finding the distance between two points.

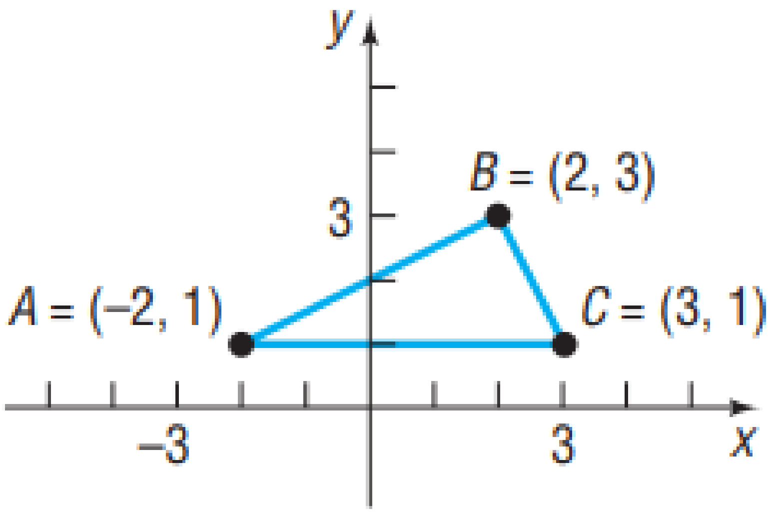
Consider the three points A = (-2,1), B = (2,3), and C = (3,1).

(a) Plot each point and form the triangle ABC

(b) Find the length of each side of the triangle.

(c) Verify that the triangle is a right triangle.

(d) Find the area of the triangle.



Solution

- Calculate the three distances (A, B), (B,C), and (A, C)
- Show the Pythagorean theorem.
- Calculate the area of the triangle.

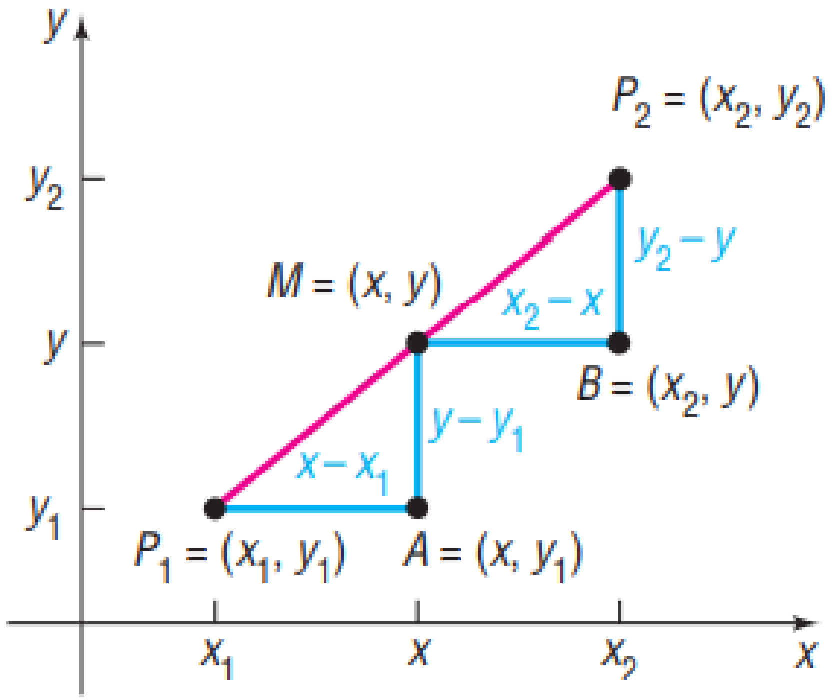
GRAPHS

Use of the Midpoint Formula

Midpoint Formula

The midpoint $M = (x, y)$ of the line segment from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ is

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \qquad \textbf{(2)}$$



Examples

Example: *Find the midpoint of a line segment*

Find the midpoint of the line segment from $P_1 = (-5,5)$ *to* $P_2 = (3,1)$*. Plot points* P_1 *and* P_2 *and their midpoint.*

Review Questions

1. On the real number line, the origin is assigned the number ….
2. If and 5 are the coordinates of two points on the real number line, the distance between these points is….
3. Use the converse of the Pythagorean Theorem to show that a triangle whose sides are of lengths 11, 60, and 61 is a right triangle

Review Questions

4. If the coordinates of a point P are in the XY-plane, then x is called the…of P, and y is the … of P.

5. The coordinate axes divide the XY-plane into four sections called…

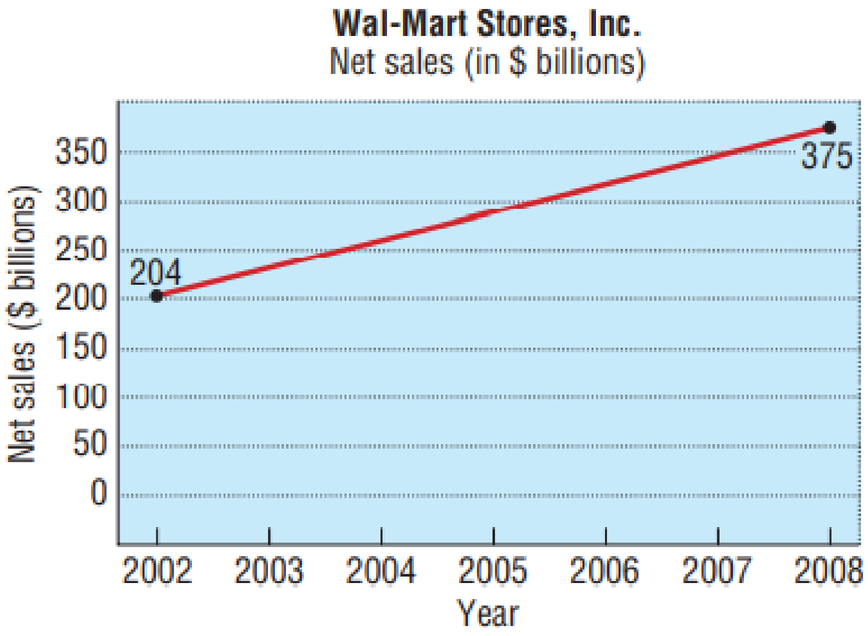
6. The distance between two points is sometimes a negative number. (TRUE/FALSE).

7. The midpoint of a line segment is found by averaging the x-coordinates and the y-coordinates of the endpoints.

Review Questions – real life

Net Sales The figure illustrates how net sales of Wal-Mart Stores, Inc., have grown from 2002 through 2008. Use the midpoint formula to estimate the net sales of Wal-Mart Stores, Inc., in 2005. How does your result compare to the reported value of $282 billion?

Source: *Wal-Mart Stores, Inc., 2008 Annual Report*



Review Questions – real life

Poverty Threshold Poverty thresholds are determined by the U.S. Census Bureau. A poverty threshold represents the minimum annual household income for a family not to be considered poor. In 1998, the poverty threshold for a family of four with two children under the age of 18 years was $16,530. In 2008, the poverty threshold for a family of four with two children under the age of 18 years was $21,834.

Assuming poverty thresholds increase in a straight-line fashion, use the midpoint formula to estimate the poverty threshold of a family of four with two children under the age of 18 in 2003. How does your result compare to the actual poverty threshold in 2003 of $18,660?

Source: *U.S. Census Bureau*

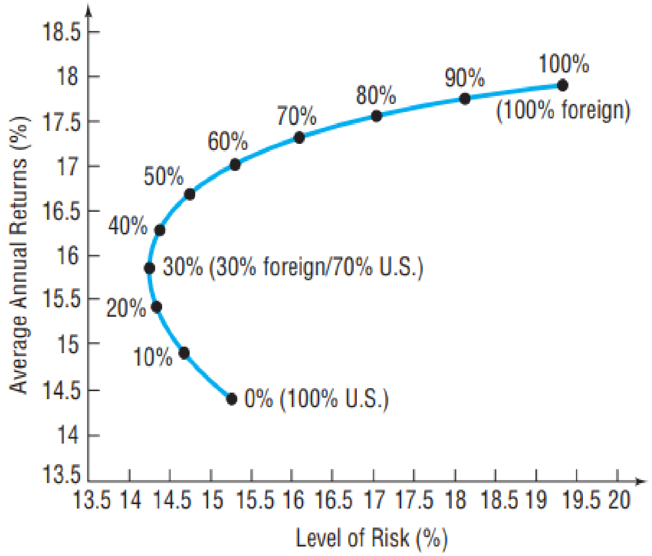
GRAPHS

The Graph of an equation

- An equation in two variables **is a statement in which two expressions (the sides)** involving x and y are equal.

- Graphs play an important role **in helping us to visualize** the relationships that exist between two variables or quantities.

Figure 11
Source: T. Rowe Price



Examples

Example: *Determining Whether a Point is on the Graph of an Equation.*

Determine if the following points are on the graph of the equation $2x-y=6$. (a) (2,3); (b) (2,-2).

Example: *Graphing an Equation by Plotting Points*

Graph the equation: $y = 2x+5$

Graphs

Intercepts from a Graph

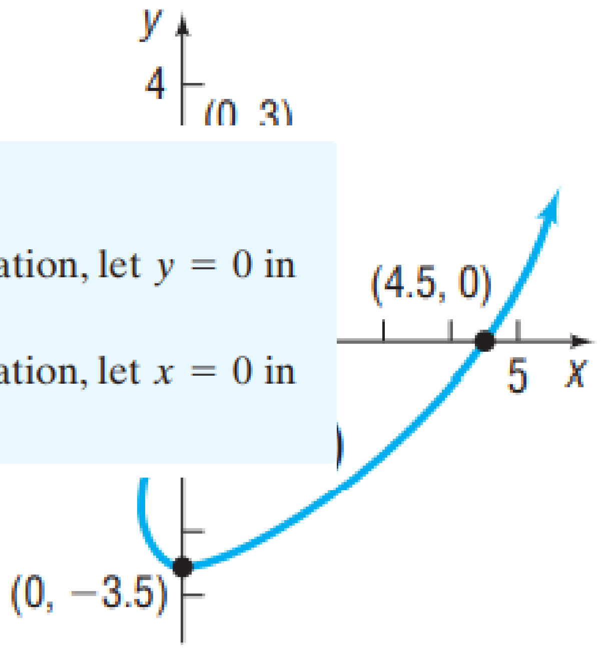
- The points at which a graph **crosses or touches the coordinate axes** are called the **intercept**

  intercept.

- The intercept

  but also in an Equation.

Find the intercepts from Graph



Procedure for Finding Intercepts

1. To find the x-intercept(s), if any, of the graph of an equation, let $y = 0$ in the equation and solve for x, where x is a real number.
2. To find the y-intercept(s), if any, of the graph of an equation, let $x = 0$ in the equation and solve for y, where y is a real number.

Examples

Example: *Finding Intercepts from an Equation.*

Find the x-intercept(s) and the y-intercepts of the graph of $y = x^2 - 4$*. Then graph the equation.*

Graphs

Symmetry with respect to the axes and the origin.

Examples

- A graph is sa
  x-axis if, for
  point (x,-y) is

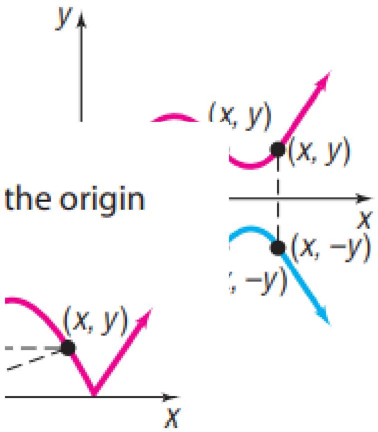
- A graph is sa
  the y-axis i
  graph, the po

- A graph is sa
  the origin if, **for every point (x,y) on the graph,**
  the point (-x,-y) is also on the graph.

Tests for Symmetry

To test the graph of an equation for symmetry with respect to the

x-Axis	Replace y by $-y$ in the equation and simplify. If an equivalent equation results, the graph of the equation is symmetric with respect to the x-axis.
y-Axis	Replace x by $-x$ in the equation and simplify. If an equivalent equation results, the graph of the equation is symmetric with respect to the y-axis.
Origin	Replace x by $-x$ and y by $-y$ in the equation and simplify. If an equivalent equation results, the graph of the equation is symmetric with respect to the origin.



Examples

Example: *Testing for Symmetry.*

Test $y = (4x^2) / (x^2 + 1)$ *for symmetry. Test it numerically but also using a graphing utility.*

Example: Using a graphing utility, please study the following key equations:

(a) $y = x^3$; (b) $x = y^2$; (c) $y = 1/x$

Graphs – Lines

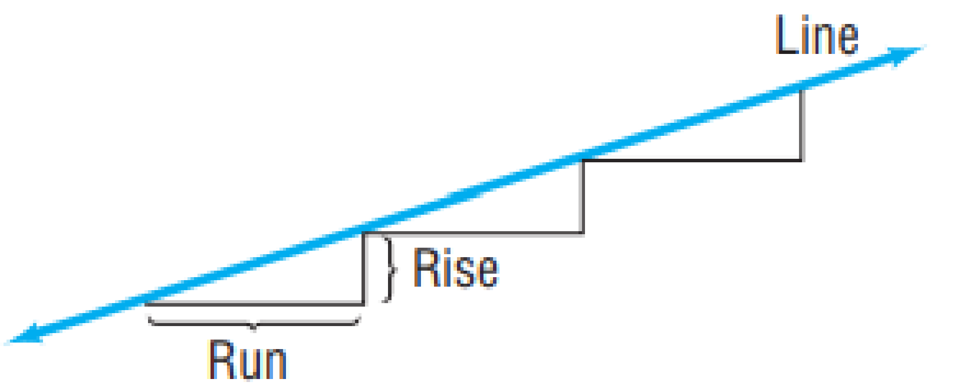
- The ratio of the **rise to run is called the slope**. It is a measure of **the steepness of the line**.

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two distinct points. If $x_1 \neq x_2$, the **slope** $\boldsymbol{m}$ of the nonvertical line L containing P and Q is defined by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad x_1 \neq x_2 \qquad \textbf{(1)}$$

If $x_1 = x_2$, L is a **vertical line** and the slope m of L is **undefined** (since this results in division by 0).

Figure 26

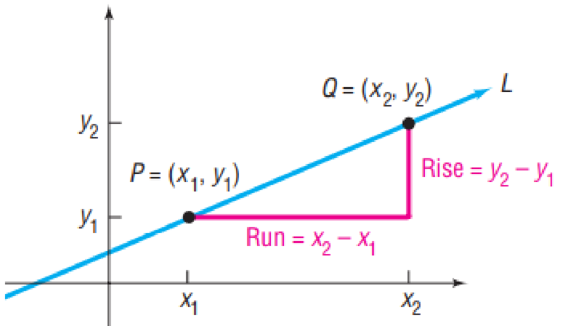


Graphs – Lines

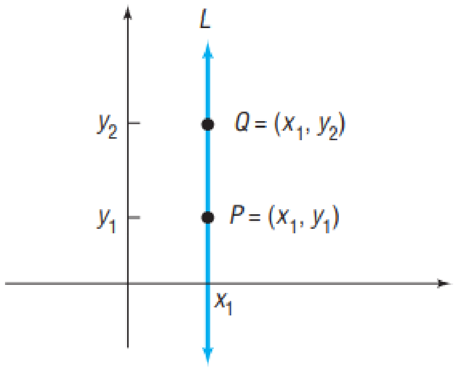
Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two distinct points. If $x_1 \neq x_2$, the **slope** $\boldsymbol{m}$ of the nonvertical line L containing P and Q is defined by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad x_1 \neq x_2 \qquad \textbf{(1)}$$

If $x_1 = x_2$, L is a **vertical line** and the slope m of L is **undefined** (since this results in division by 0).



(a) Slope of L is $m = \frac{y_2 - y_1}{x_2 - x_1}$



(b) Slope is undefined; L is vertical

Examples

Example: *Finding and Interpreting the Slope of a Line Given two Points.*

Calculate the slope m of the line containing points (1,2) and (5, -3).

Compute the slopes of the lines L_1: $P = (2,3)$ *and* $Q_1 = (-1, -2)$ / L_2 : $P = (2,3)$ *and* $Q_2 = (3, -1)$

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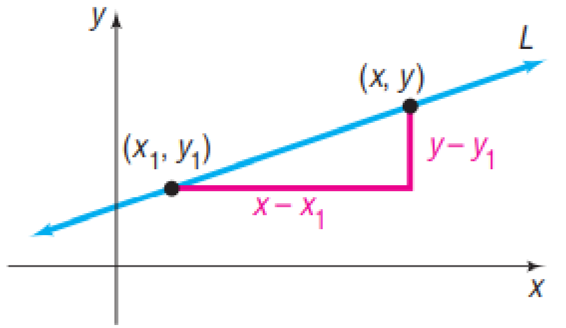
Examples

Example: *Graphing a Line Given a Point and a Slope.*

Draw a graph of the line that contains points (3,2) and has a slope of (a) ¾ and (b) -4/5.

Graphs – Lines

Figure 35



THEOREM

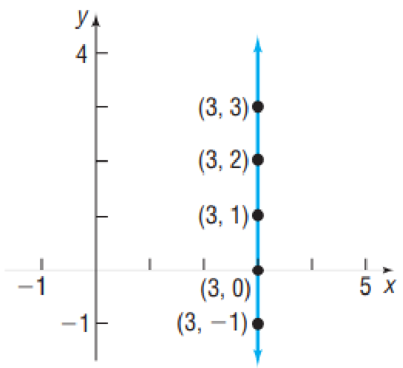
Point–Slope Form of an Equation of a Line

An equation of a nonvertical line with slope m that contains the point (x_1, y_1) is

$$y - y_1 = m(x - x_1) \qquad (2)$$

Graphs – Lines

Figure 34



THEOREM

Equation of a Vertical Line

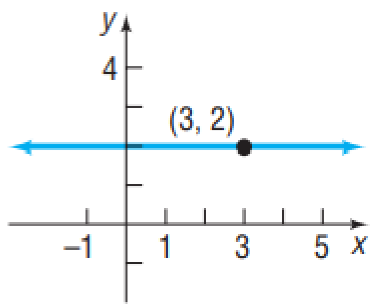
A vertical line is given by an equation of the form

$$x = a$$

where a is the x-intercept.

Graphs – Lines

Figure 37



THEOREM

Equation of a Horizontal Line

A horizontal line is given by an equation of the form

$$y = b$$

where b is the y-intercept.

Examples

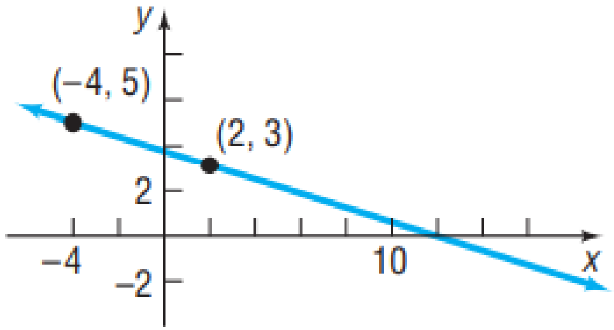
Example: *Using the Point – Slope Form of a Line.*

An equation of the line with slope 4 containing the point (1,2) can be found using the point-slope form.

Example: *Finding the Equation of a Horizontal Line.*

Find the equation of the horizontal line containing the point (3,2).

Graphs – Lines



THEOREM

Slope–Intercept Form of an Equation of a Line

An equation of a line with slope m and y-intercept b is

$$y = mx + b \qquad (3)$$

The question in the classroom: What is the physical meaning of m and b?

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Examples

Example: *Finding the slope and y-intercept.*

Find the slope m and y-intercept b of the equation $2x+4y=8$. Graph the equation.

Graphs – Lines

Graph Lines Written in general Form Using Intercepts

DEFINITION

The equation of a line is in **general form*** when it is written as

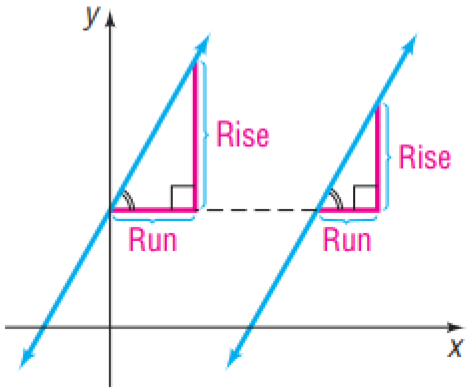
$$Ax + By = C \qquad \textbf{(4)}$$

where A, B, and C are real numbers and A and B are not both 0.

Example: Graph the equation 2x + 4y = 8 by finding its intercepts.

Graphs – Lines

Finding Equations of two parallel lines.



THEOREM Criterion for Parallel Lines

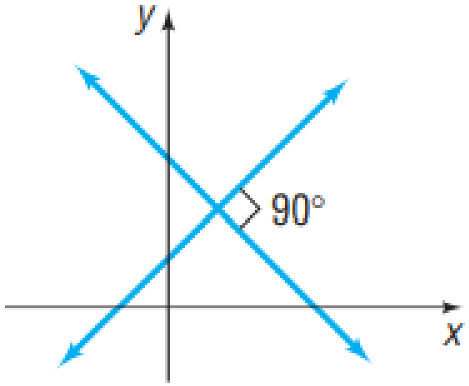
Two nonvertical lines are parallel if and only if their slopes are equal and they have different *y*-intercepts.

Example: Show that the lines given by the following equations are parallel:

L_1: 2x + 3y = 6 and L_2: 4x + 6y = 0

Graphs – Lines

Finding Equations of two Perpendicular lines.



THEOREM

Criterion for Perpendicular Lines

Two nonvertical lines are perpendicular if and only if the product of their slopes is -1.

Example: if a line has a slope 3/2, any line having a slope of -2/3 is perpendicular to it.

Example: Find an equation of the line that contains the point (1, -2) and is perpendicular to the line

$x + 3y = 6$

Review Questions – real life

Electricity Rates in Florida Florida Power & Light Company supplies electricity to residential customers for a monthly customer charge of \$5.69 plus 8.48 cents per kilowatt-hour for up to 1000 kilowatt-hours.

(a) Write a linear equation that relates the monthly charge C, in dollars, to the number x of kilowatt-hours used in a month, $0 \leq x \leq 1000$.

(b) Graph this equation.

(c) What is the monthly charge for using 200 kilowatt-hours?

(d) What is the monthly charge for using 500 kilowatt-hours?

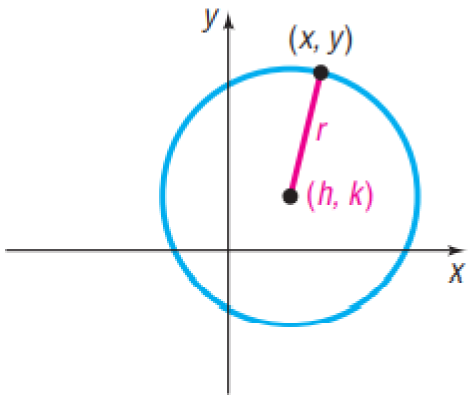
(e) Interpret the slope of the line.

Source: Florida Power & Light Company, February, 2010.

Graphs - Circles

DEFINITION

A **circle** is a set of points in the xy-plane that are a fixed distance r from a fixed point (h, k). The fixed distance r is called the **radius,** and the fixed point (h, k) is called the **center** of the circle.



DEFINITION

The **standard form of an equation of a circle** with radius r and center (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \textbf{(1)}$$

DEFINITION

If the radius $r = 1$, the circle whose center is at the origin is called the **unit circle** and has the equation

$$x^2 + y^2 = 1$$

DEFINITION

When its graph is a circle, the equation

$$x^2 + y^2 + ax + by + c = 0$$

is referred to as the **general form of the equation of a circle.**

Graphs – Circles.

Example: *Writing the Standard Form of the Equation of a Circle*

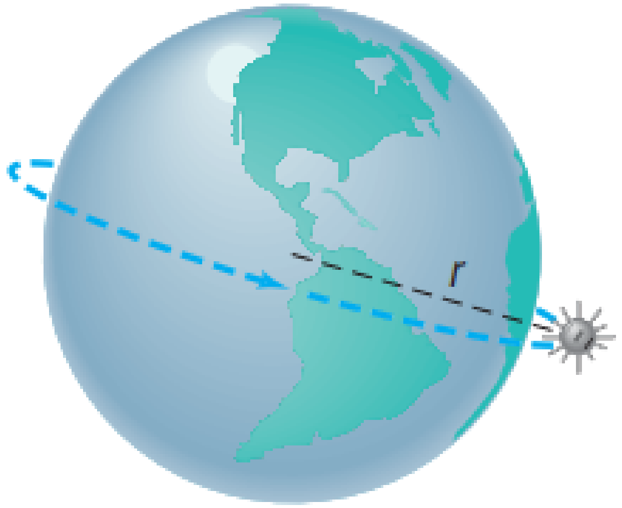
Write the standard form of the circle equation with radius 5 and center (-3, 6).

Example: Graph the equation: $(x + 3)^2 + (y - 2)^2 = 16$.

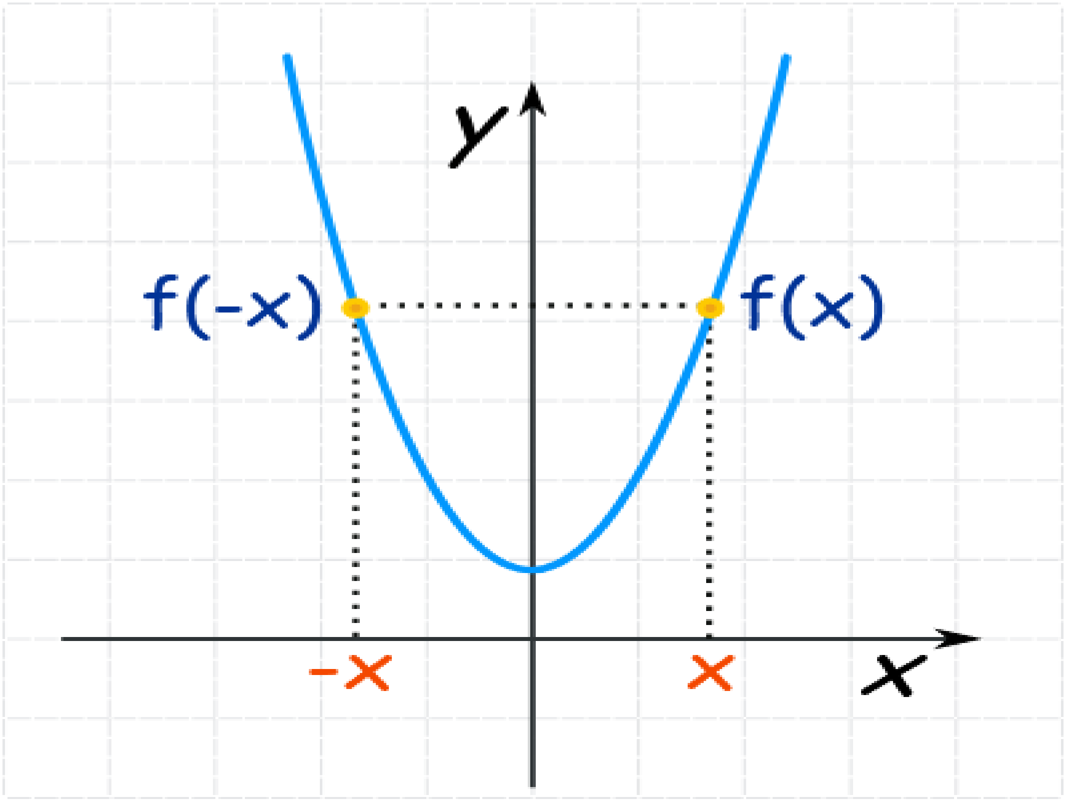
Example: For the circle $(x + 3)^2 + (y - 2)^2 = 16$, find the intercepts, if any, of its graph.

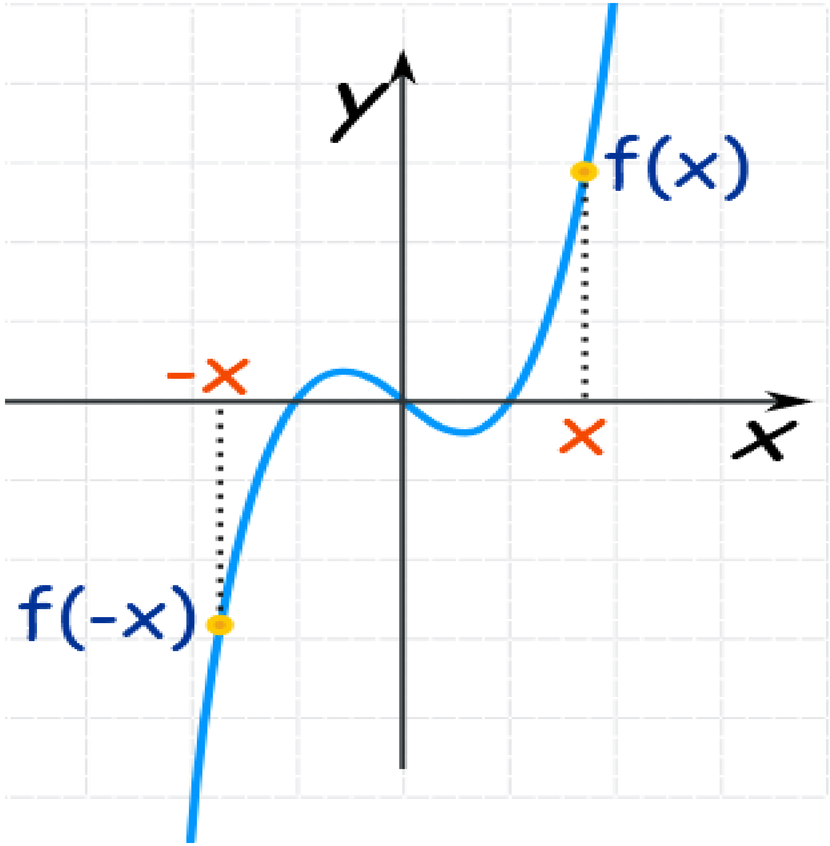
Review Questions – real life

Weather Satellites Earth is represented on a map of a portion of the solar system so that its surface is the circle with equation $x^2 + y^2 + 2x + 4y - 4091 = 0$. A weather satellite circles 0.6 unit above Earth with the center of its circular orbit at the center of Earth. Find the equation for the orbit of the satellite on this map.

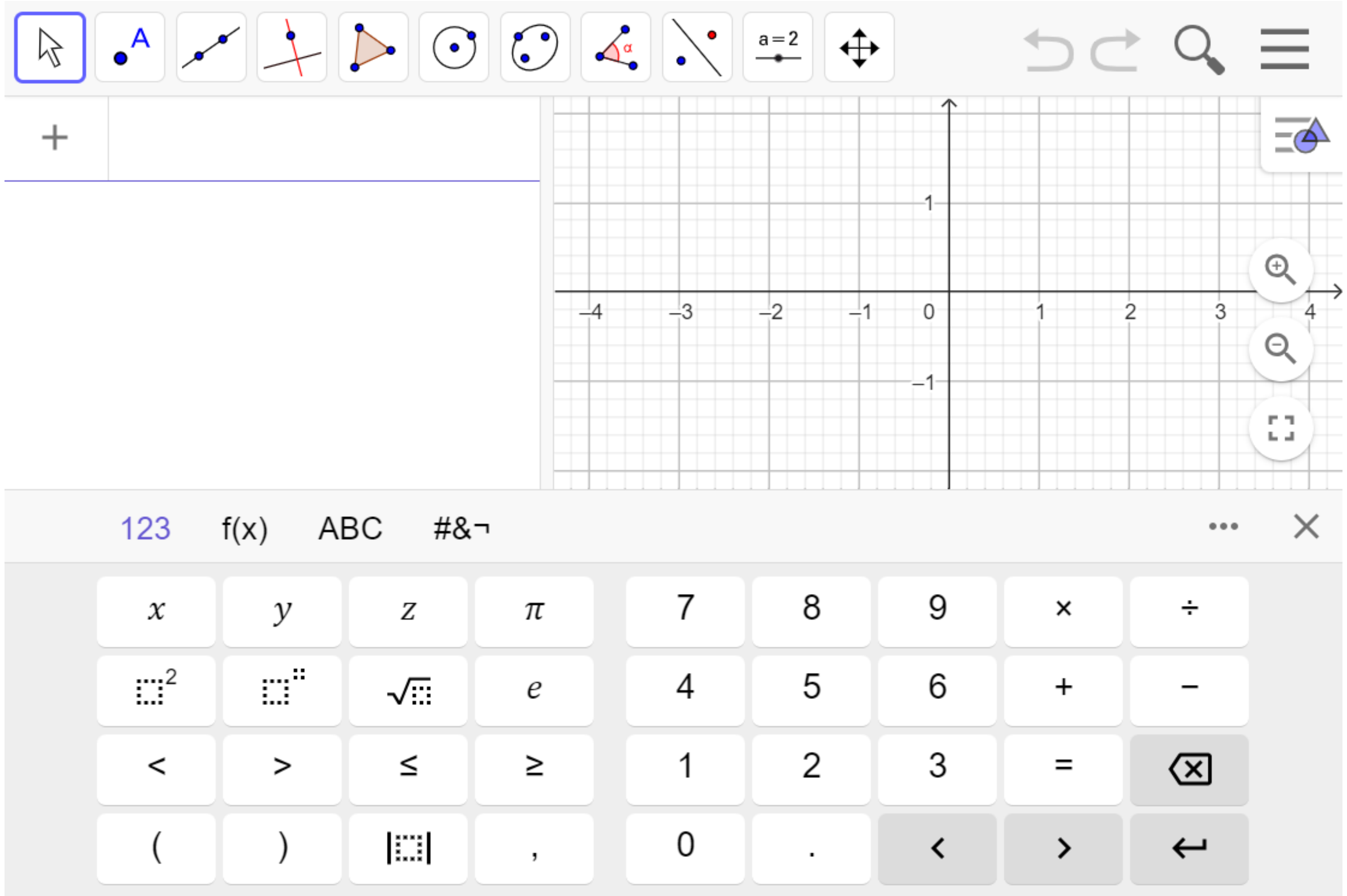


Even and Odd functions

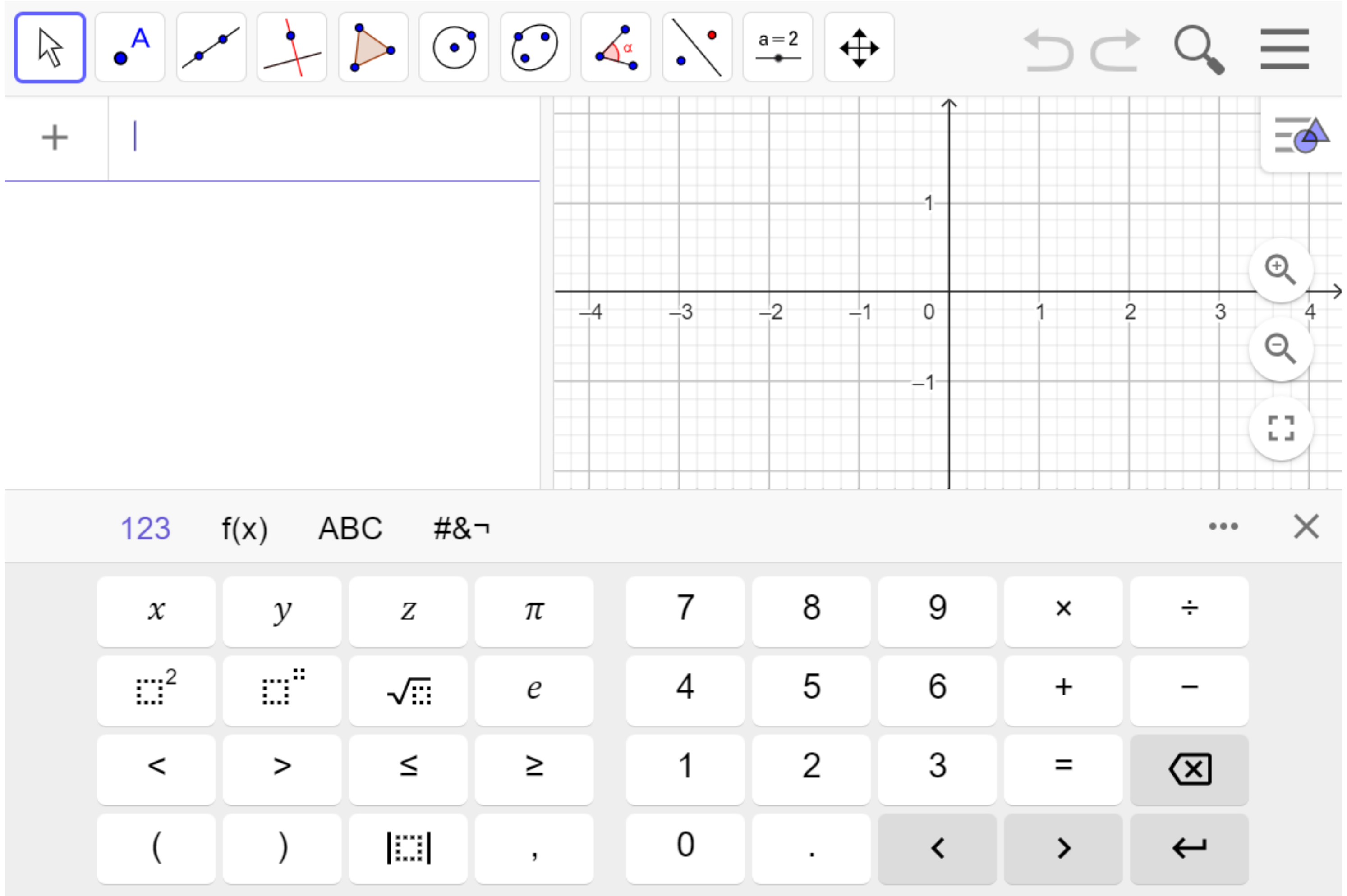




Even and Odd functions



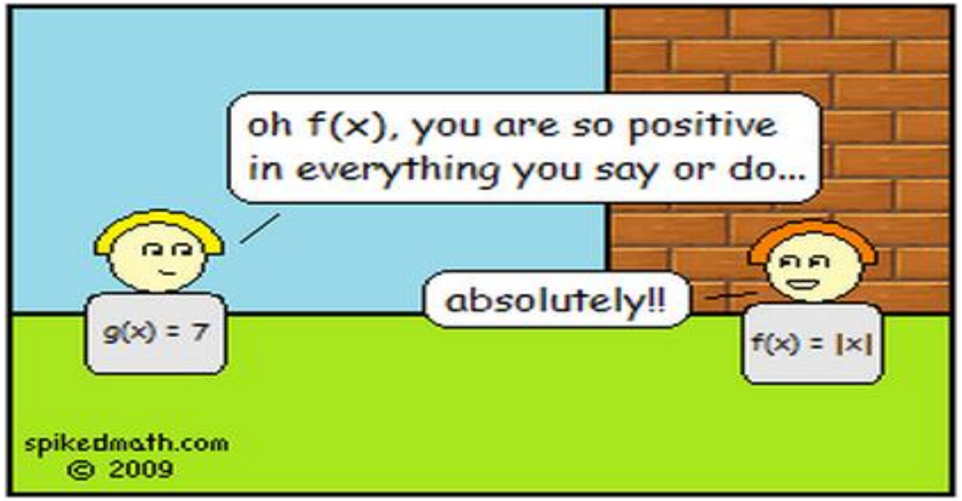
Even and Odd functions

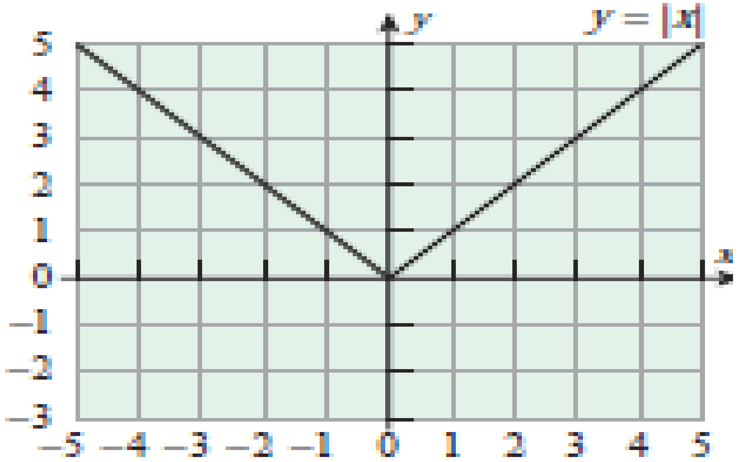


The absolute value function

$$f(x) = |x|$$

$$y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$





The absolute value function

$$|a| = 0 \Leftrightarrow a = 0$$

$$|a| = |-a|$$

$$|a| = a \Leftrightarrow a \geq 0$$

$$|a| \neq a \Leftrightarrow a < 0$$

$$|a| = -a \Leftrightarrow a \leq 0$$

$$|a| \neq -a \Leftrightarrow a > 0$$

The absolute value function

Properties

1. $|ab| = |a||b|$
2. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \quad (b \neq 0)$
3. $||a| - |b|| \leq |a + b| \leq |a| + |b|$(trigonometric inequality)
4. $|x| = a \Leftrightarrow x = -a \; or \; x = a$
5. $|x| < a \Leftrightarrow -a < x < a$
6. $|x| > a \Leftrightarrow x < -a \; or \; x > a$

Polynomial Functions

- **What is a polynomial function?** A polynomial function is a function that can be constructed from a variable and a set of numbers, using only addition and multiplication

- The **standard form** of a polynomial function is:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_o$$

where the **a_i are called the coefficients** of the polynomial and the term on **the highest power defines the order of the polynomial**

Polynomial Functions

- **The importance of the polynomial degree is crucial** since determines for x >>0 or x <<0 the behaviour of the polynomial ($f(x) \sim a_n x^n$)

- **An example:**

$$p(x) = 2x^3 + 8x^2 - 13x$$

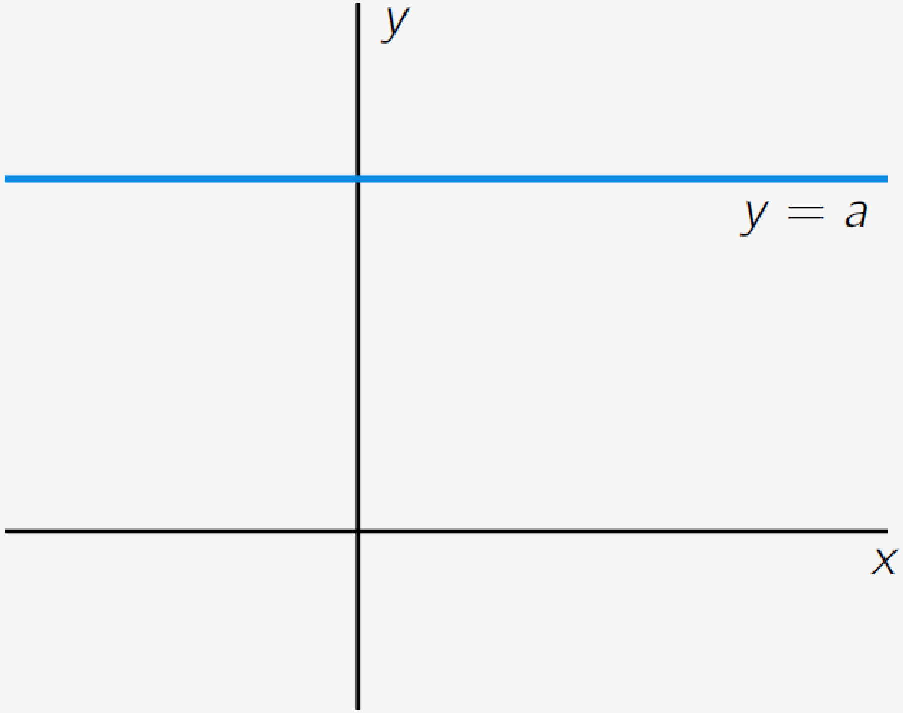
x	$2x^3$	$8x^2$	-13x	*p*(x)
-1	-2	8	13	19
-10	-2,000	800	130	-1,070
-100	-2,000,000	80,000	1,300	-1,918,700
-1,000	-2,000,000,000	8,000,000	13,000	-1,991,987,000

$$p(x) \sim 2x^3$$

Polynomial Functions

- In this pre-calculus module the following polynomial functions will be studied:
  - The **zeroth order** polynomial function or otherwise called **the constant function:** y(x) = a
  - The **1st order** polynomial function or otherwise the **linear function:** y(x) = ax+b
  - The **2nd order** polynomial function or otherwise the **quadradic function**: $y(x)=ax^2+bx+c$

Polynomial Functions



Standard form:

$f(x) = a$

$= 1$

Graph:

horizontal line

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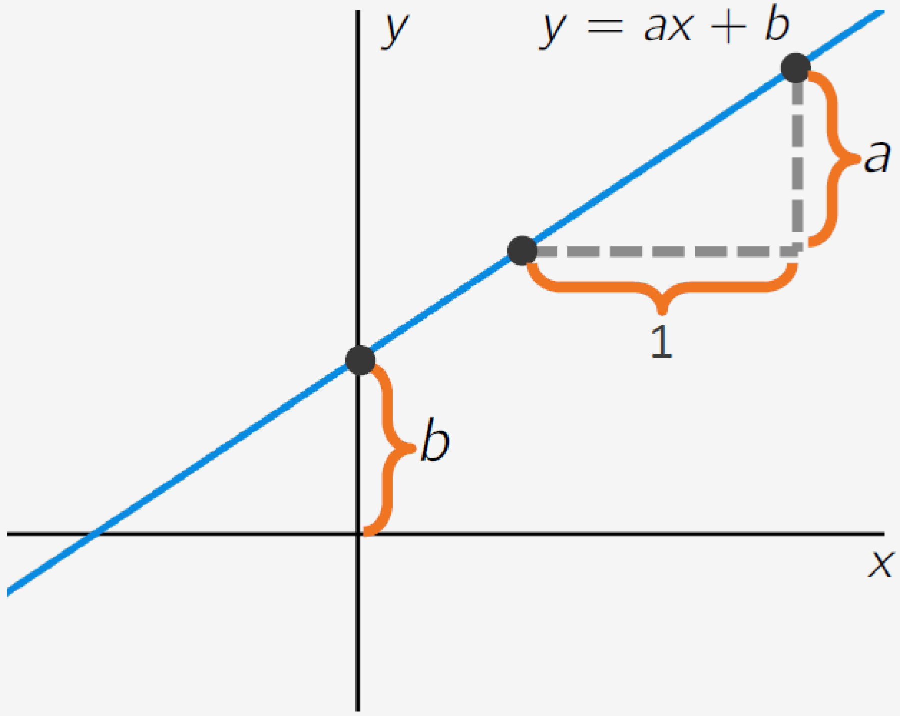
Polynomial Functions

Degree 1: linear functions



Standard form:

$f(x) = ax + b$

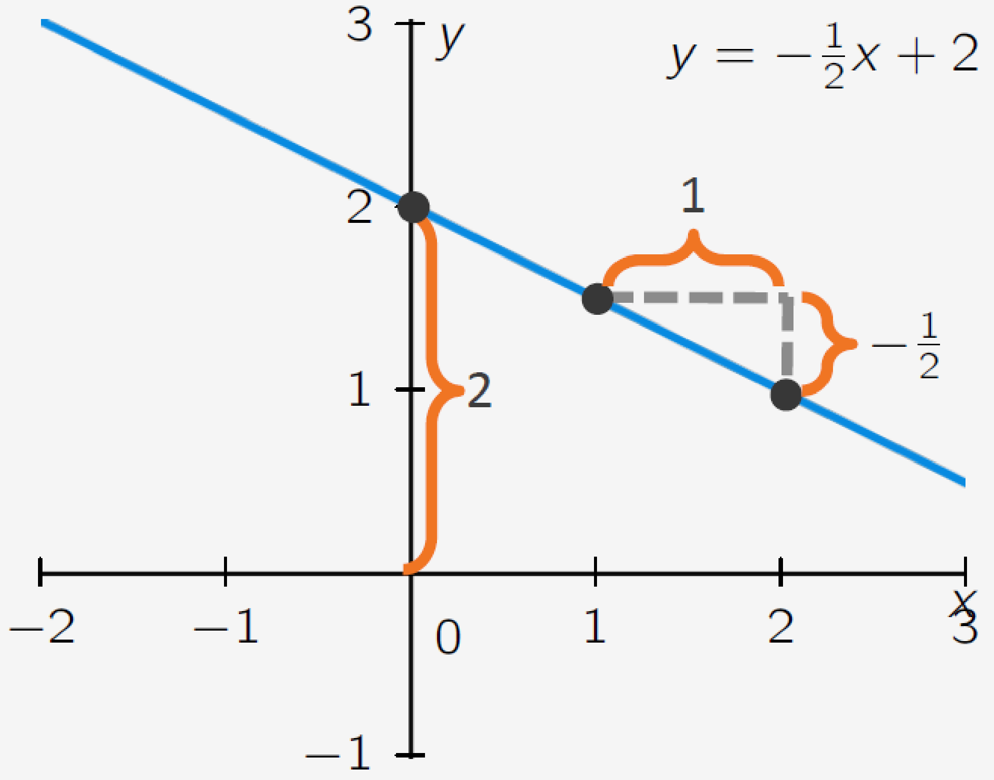
Graph:

straight line

a = slope

b = y - intercept

Polynomial Functions



Example:

$f(x) = -\frac{1}{2}x + 2$

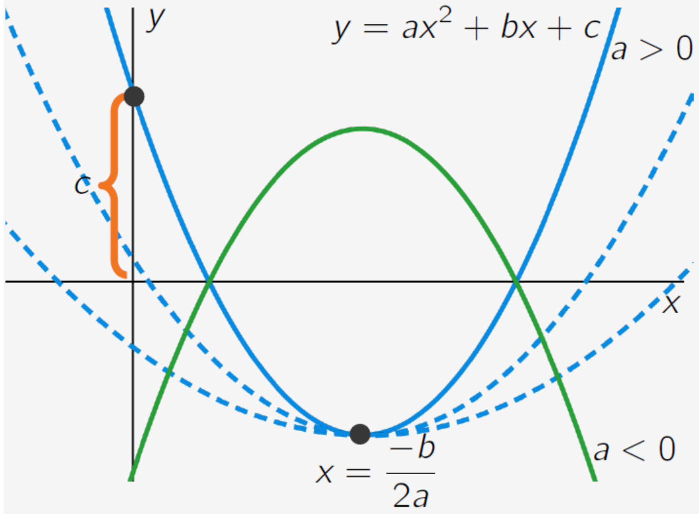
Graph:

slope = $-\frac{1}{2}$

$y-$ intercept = 2

Polynomial Functions

Degree 2: quadratic functions



Standard form:

$f(x) = ax^2 + bx + c$

Graph: parabola

a = wideness and orientation

$\frac{-b}{2a}$ = x-position vertex

c = y - intercept

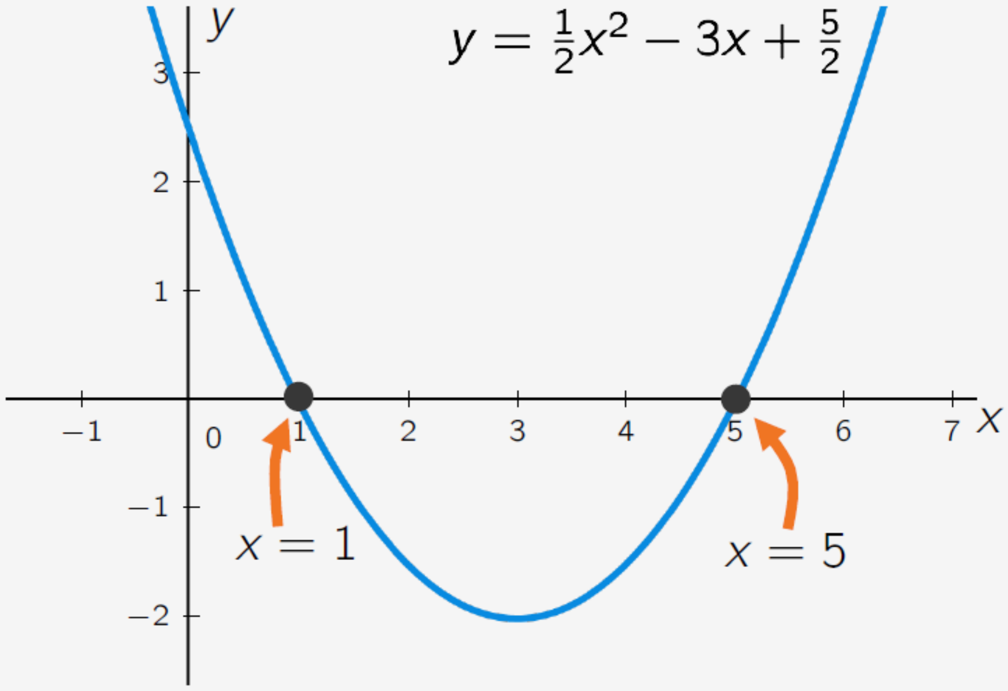
$h(x) \approx 0.00037x^2 - 0.475x + 230$ above sealevel in m

x in m from left pillar

As the **a gets higher values** the graph the narrower is made

Polynomial Functions

Degree 2: alternate forms



Factorized form:

$f(x) = a(x - p)(x - q)$

Graph intersects x-axis at $x = p$ and $x = q$

Example:

$f(x) = \frac{1}{2}(x - 1)(x - 5)$

Polynomial Functions

But if you have a quadratic function that is in standard form $ax^2 + bx + c$, how to factor it? In general, it is not easy without actually solving the equation $ax^2 + bx + c = 0$ to find p and q. However, in some cases you can find a factorization by making an educated guess. For example, suppose we want to find p and q such that

$$2x^2 - 8x + 6 = a\left(x - p\right)\left(x - q\right).$$

Let's rewrite both sides:

$$2\left(x^2 - 4x + 3\right) = a\left(x^2 - \left(p + q\right)x + pq\right).$$

Both sides are equal precisely if $a = 2$, $p + q = 4$ and $pq = 3$. If you think about it, it is easy to see that $p = 1$ and $q = 3$ will do. So we find:

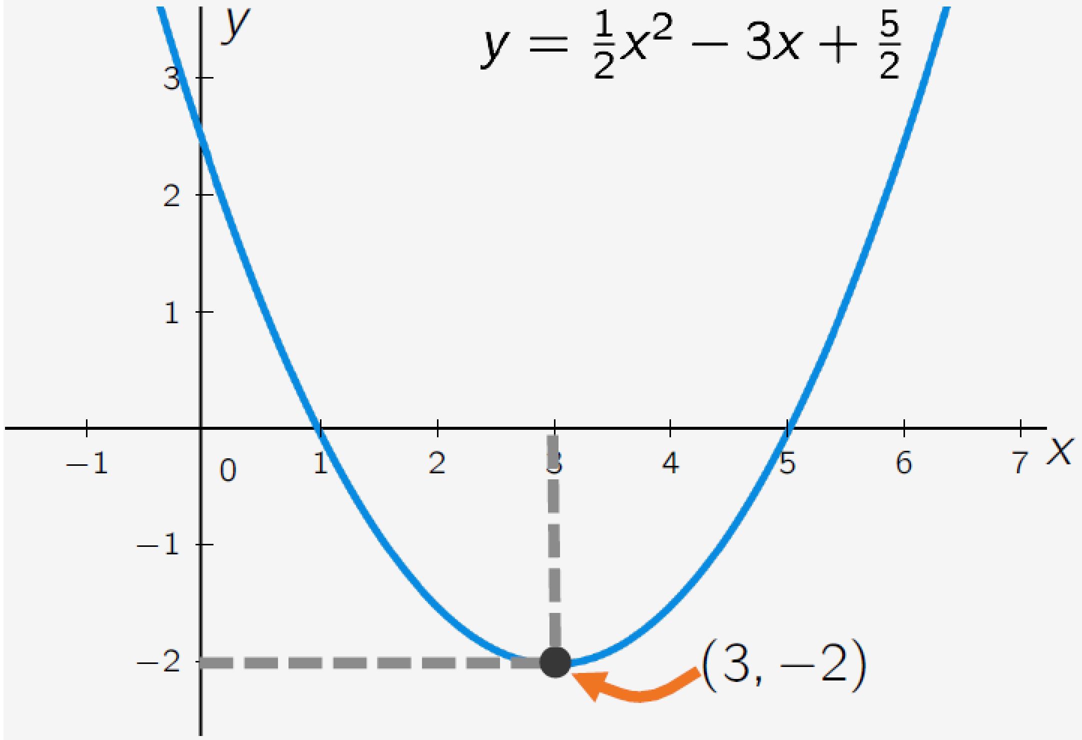
$$2x^2 - 8x + 6 = 2\left(x - 1\right)\left(x - 3\right).$$

In general, factoring $ax^2 + bx + c$ to $a\left(x - p\right)\left(x - q\right)$ amounts to solving:

$$p + q = -\frac{b}{a} \quad \text{and} \quad p \cdot q = \frac{c}{a}.$$

Polynomial Functions

Degree 2: alternate forms



Complete-square form:

$f(x) = a(x - r)^2 + s$

Vertex at (r, s)

Example:

$f(x) = \frac{1}{2}(x - 3)^2 - 2$

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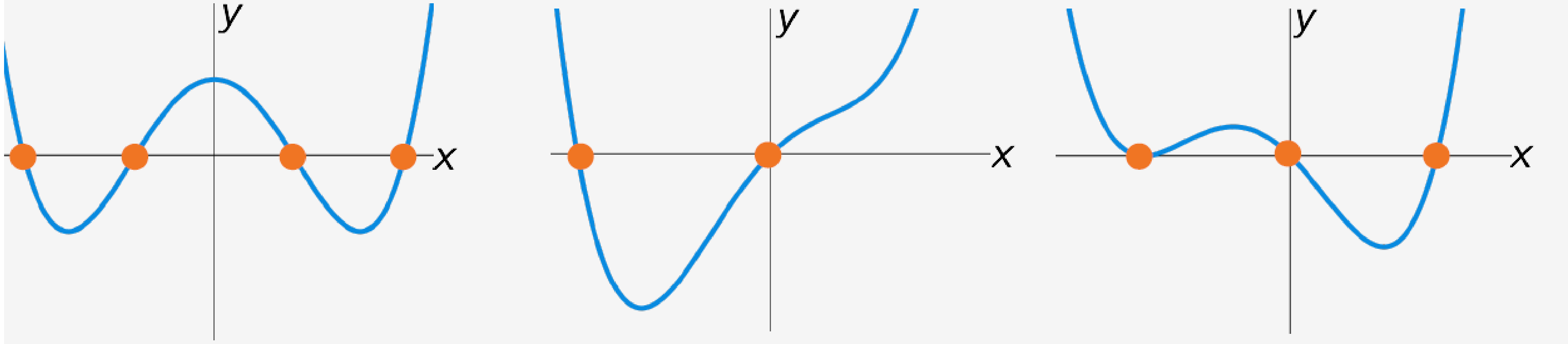
Polynomial Functions

Degree n: $f(x) = a_n x^n + \ldots + a_0$

Graph intersects x-axis in *n* points or fewer.

$y = x^4 - 2x^2 + \frac{1}{2}$ $\quad$ $y = x^4 - x^2 + x$ $\quad$ $y = x^4 + x^3 - x^2 - x$



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Polynomial Functions

- If you want to learn more **about polynomials** please check the following video lectures:

  https://www.youtube.com/watch?v=tIbqykYUZNM&feature=youtu.be

Rational Functions

- **What is a rational function?** A rational function is a function that can be constructed from a variable and a set of numbers, using addition, multiplication and division

$R(x) = \frac{P(x)}{Q(x)}$ for polynomials $P(x), Q(x)$

Domain: excludes points with $Q(x) = 0$

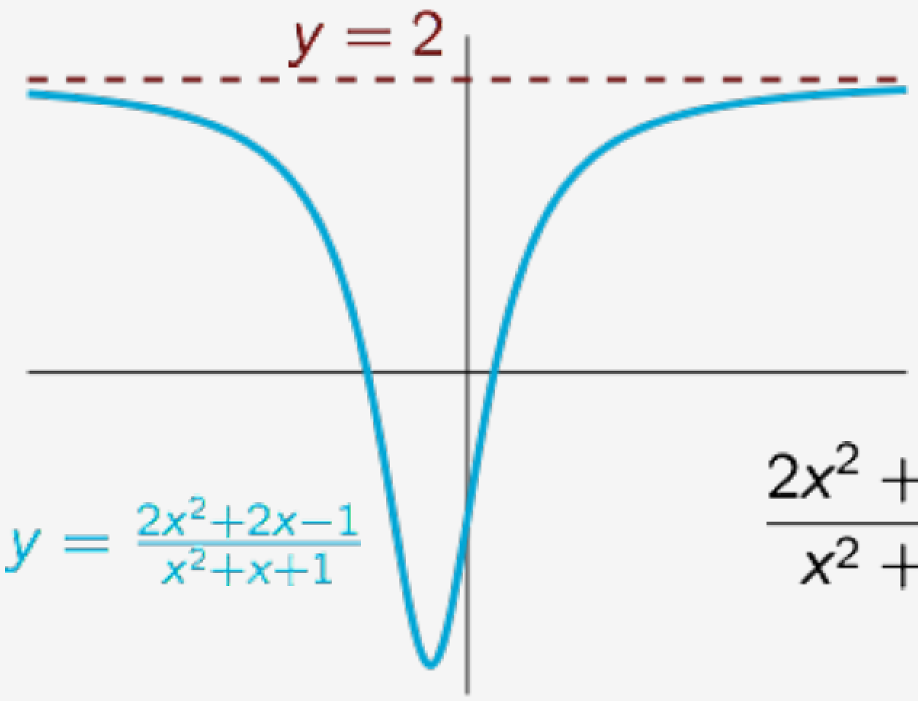
Examples:

$$\frac{x^2+3x+2}{1} \qquad \frac{1}{x^2+2x+3} \qquad 1+\frac{1}{x}+\frac{1}{x^2}=\frac{x^2+x+1}{x^2}$$

Rational Functions

Horizontal asymptotes

$$R(x) = \frac{P(x)}{Q(x)}$$



Horizontal asymptote:
if $\deg(P) \leq \deg(Q)$

Determine location by dividing by $x^{\deg(Q)}$

$$\frac{2x^2+2x-1}{x^2+x+1} = \frac{2+2\frac{1}{x}-\frac{1}{x^2}}{1+\frac{1}{x}+\frac{1}{x^2}} \approx \frac{2}{1} = 2$$

For large x

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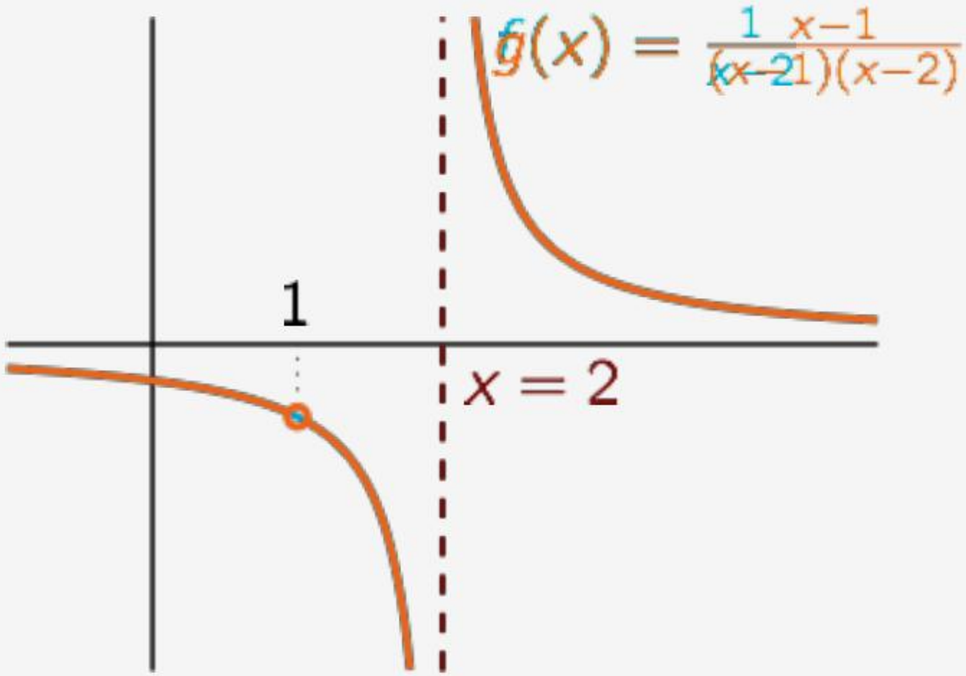
Rational Functions

Vertical asymptotes

$$R(x) = \frac{P(x)}{Q(x)}$$



Vertical asymptotes: can only occur at $Q(x) = 0$

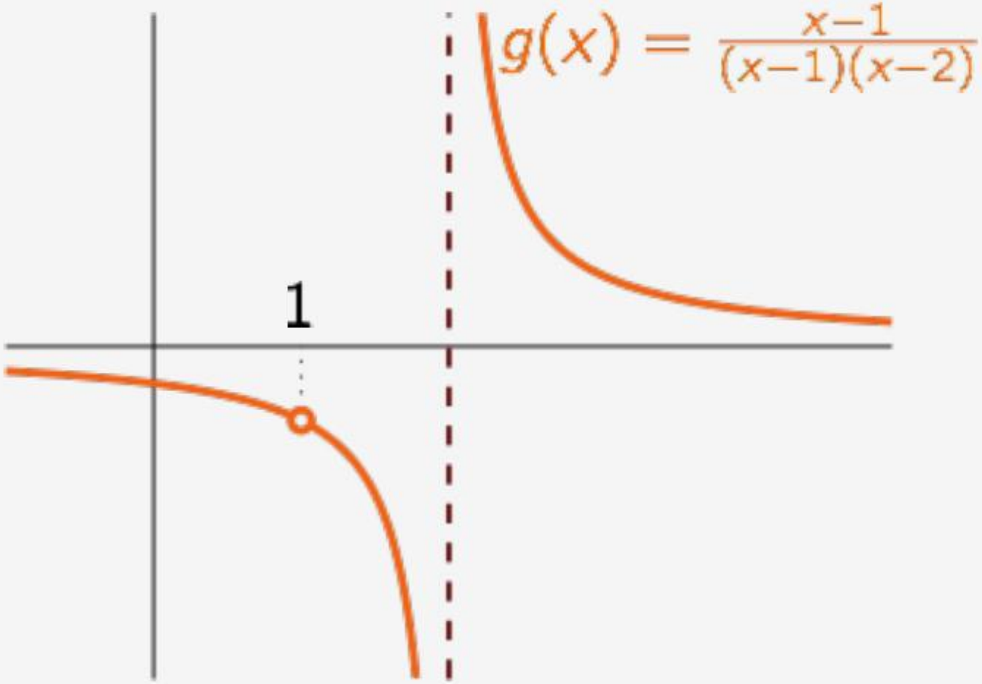
$$f(x) = \frac{1}{x-2}$$

$$g(x) = \frac{x-1}{(x-1)(x-2)}$$

$$g(x) = f(x) \text{ for } x \neq 1$$

Rational Functions

Zeros of rational functions



$$\frac{P(x)}{Q(x)} = 0 \text{ whenever}$$

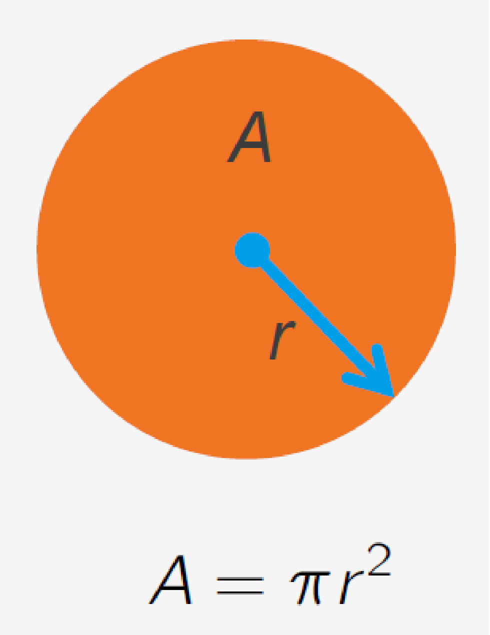
$$P(x) = 0 \text{ and } Q(x) \neq 0$$

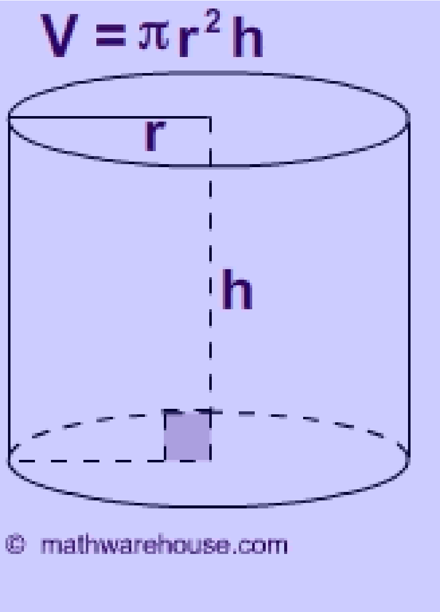
Power Functions

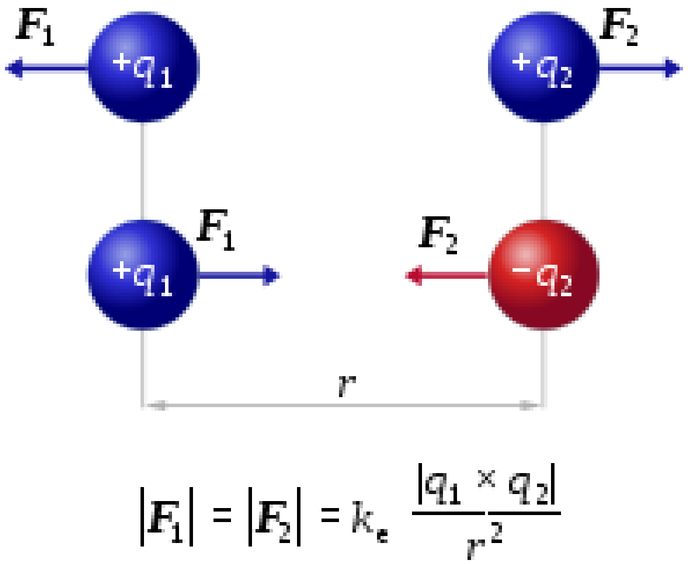
- The power functions result from the **multiplication of monomials e.g. $x^2 = x*x$**
- The general form of a power function is the following: **$f(x) = x^a$, where a is an integer (positive or negative) and is called the exponent**
- Power functions **with positive integer exponent demonstrate the same symmetries, follow the same calculation rules as these ones with negative integer exponent…..but have completely different properties**
- In this section we will study power functions with positive, negative integer and non integer exponents

Power Functions

- **The power functions in real life are very useful** since can be used to calculate the surface of a cycle, the volume of a cylinder or the force between two charged







Power Functions

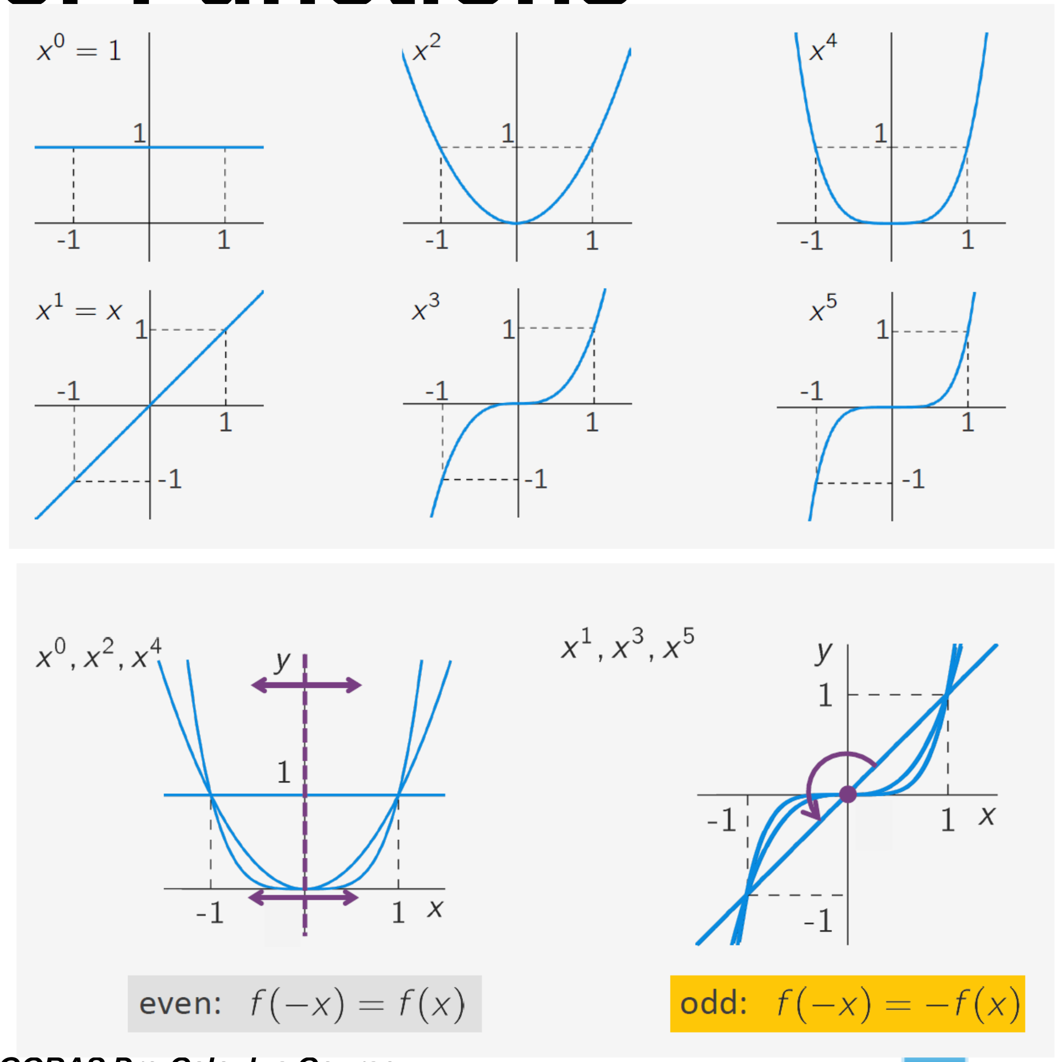
$f(x) = x^a$ for nonnegative integer a

Rules of calculation

$$x^a x^b = x^{a+b}$$

$$(x^a)^b = x^{a \cdot b}$$

$$(x\,y)^a = x^a\,y^a$$



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Power Functions

$f(x) = x^a$, a constant

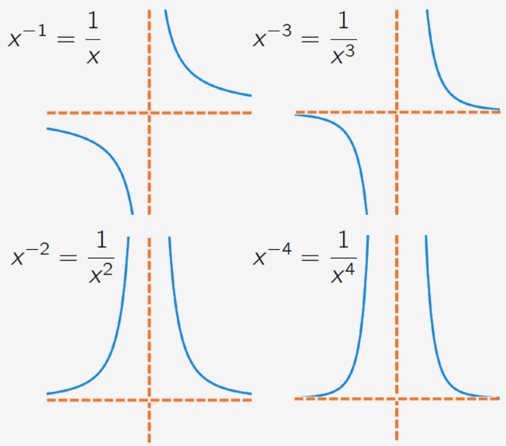
Rules of calculation

$$x^a x^b = x^{a+b}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$(x^a)^b = x^{a \cdot b}$$

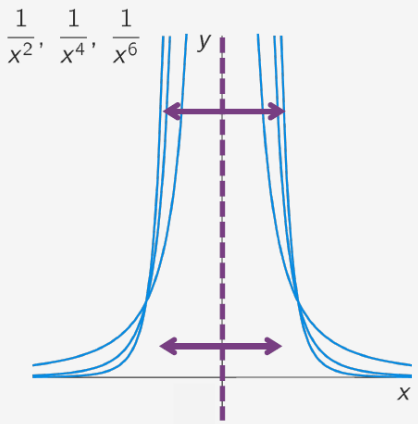
$$(x\,y)^a = x^a\,y^a$$



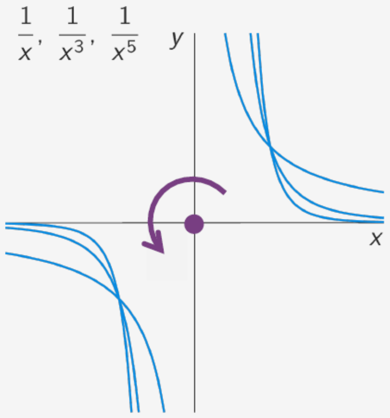
$f(x) = x^a$, a negative

Graph has

- vertical asymptote at $x = 0$
- horizontal asymptote at $y = 0$



even: $f(-x) = f(x)$



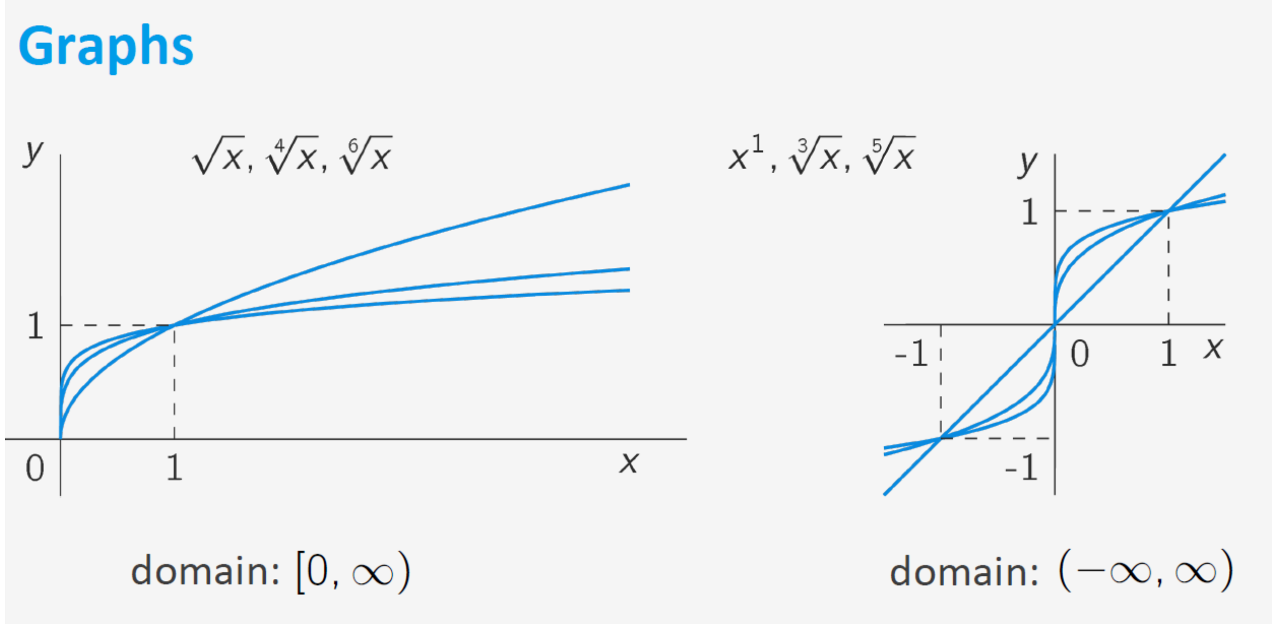
odd: $f(-x) = -f(x)$

Power Functions

- In this case we have **a do not be an integer** e.g. a = 1/2 , how do we define $x^{1/2}$?

- **In general if a = 1/n** with n = 1,2,3….then:

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$



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Power Functions

Examples:

- $x^{\frac{2}{3}} = (x^2)^{\frac{1}{3}} = \sqrt[3]{x^2}$

  $x^{\frac{2}{3}} = (x^{\frac{1}{3}})^2 = (\sqrt[3]{x})^2$

- $x^{-\frac{5}{2}} = \left(\frac{1}{x^5}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{x^5}}$

  $x^{-\frac{5}{2}} = (\sqrt{x})^{-5} = \frac{1}{(\sqrt{x})^5}$

$$x^{\frac{p}{q}} = \sqrt[q]{x^p} = (\sqrt[q]{x})^p$$

$$x^{-\frac{p}{q}} = \frac{1}{\sqrt[q]{x^p}} = \frac{1}{(\sqrt[q]{x})^p}$$

Power Functions

For which values of x is x^a defined?

	$a \geq 0$	$a < 0$
In general	$x \geq 0$	$x > 0$
Integer a … or $a = \frac{p}{q}$ with q odd	all x	$x \neq 0$

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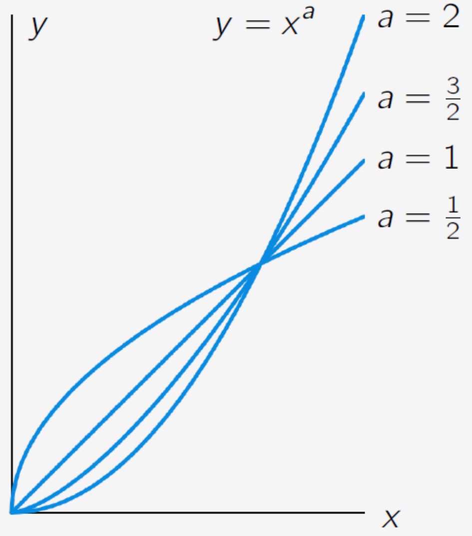
Power Functions

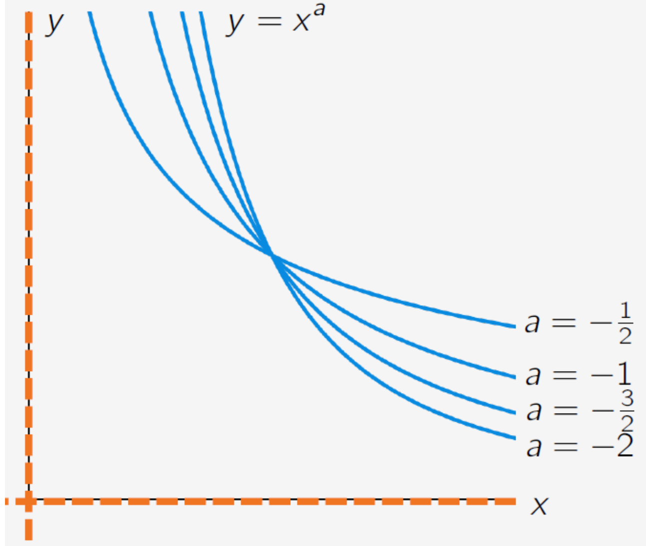
General properties



Graphs $(x > 0)$

- $a > 0$: increasing

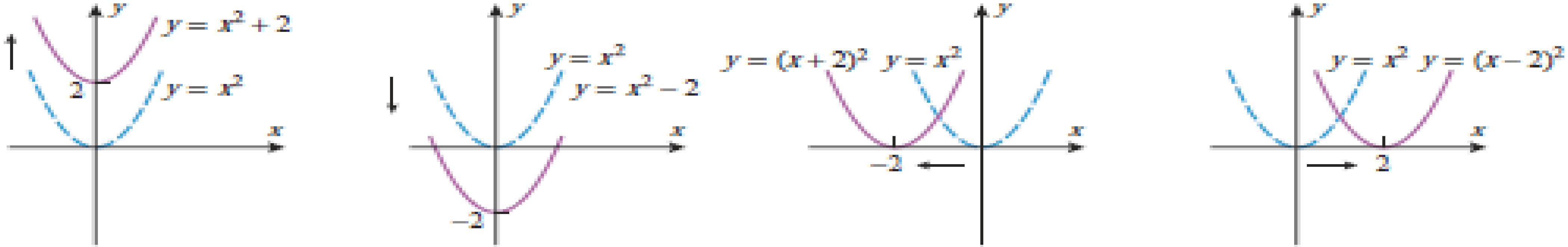
General properties

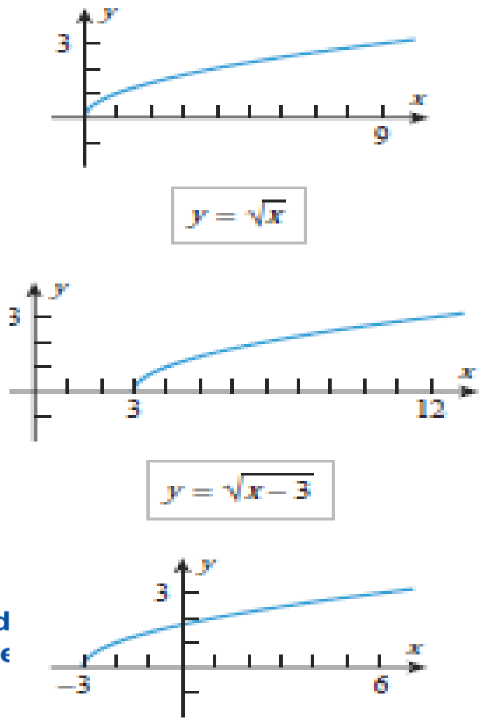


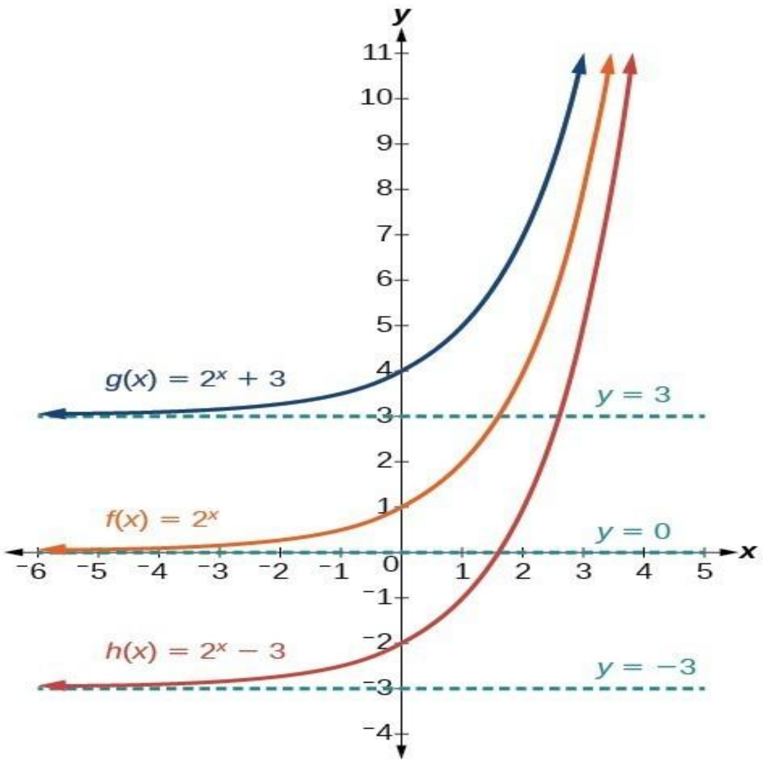
Graphs

- $a > 0$: increasing for $x > 0$
- $a < 0$: decreasing for $x > 0$
  asymptotes at $x = 0$ and $y = 0$

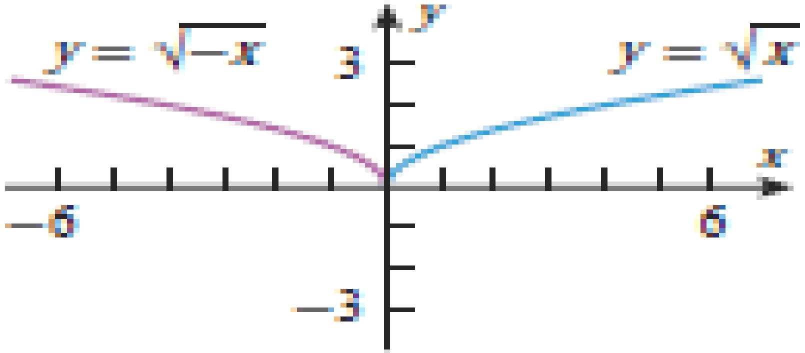
Translations

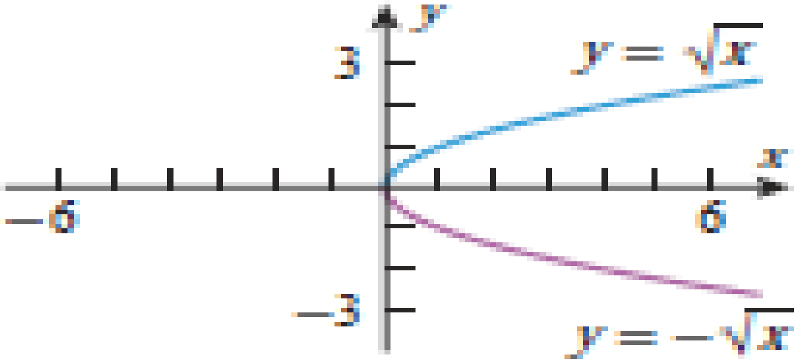


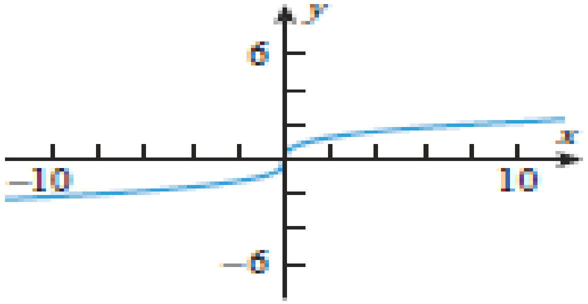




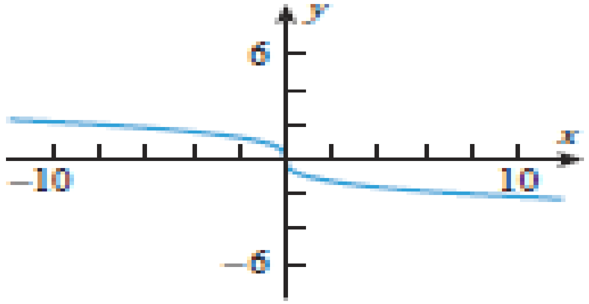
Reflections



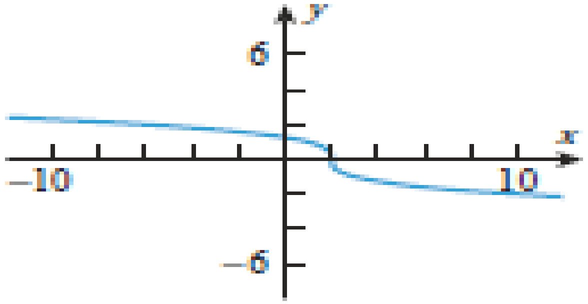




$y = \sqrt[3]{x}$



$y = \sqrt[3]{-x}$



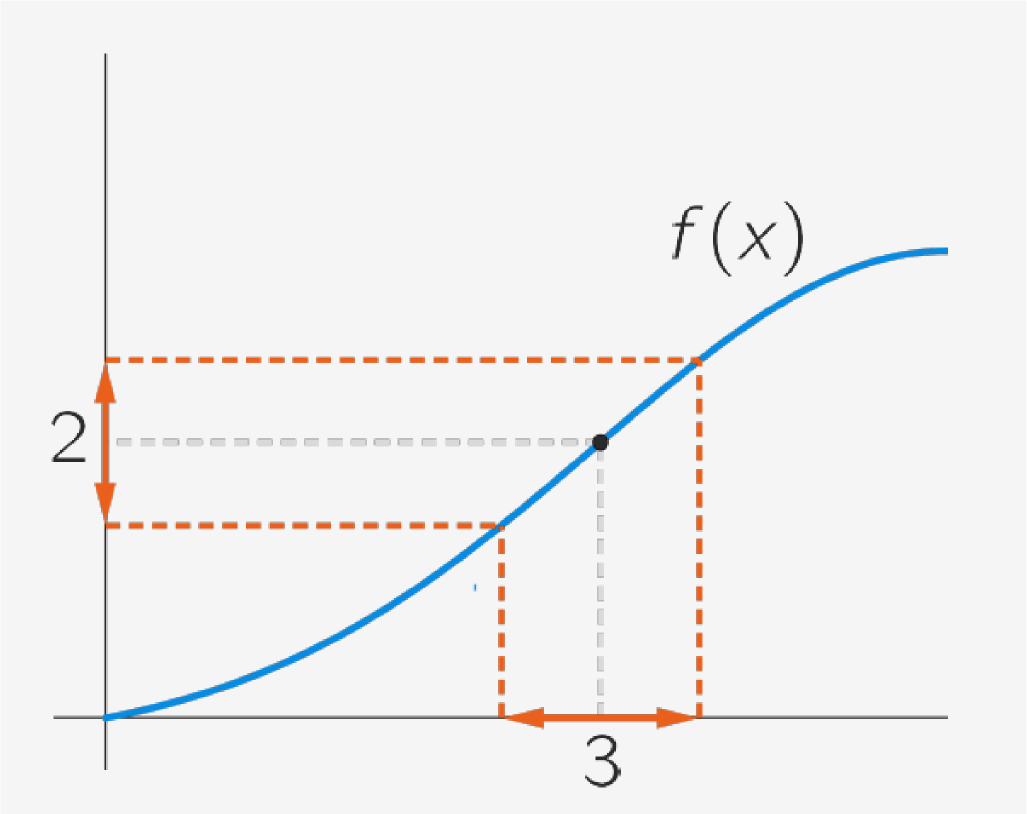
$y = \sqrt[3]{2 - x}$

Continuity

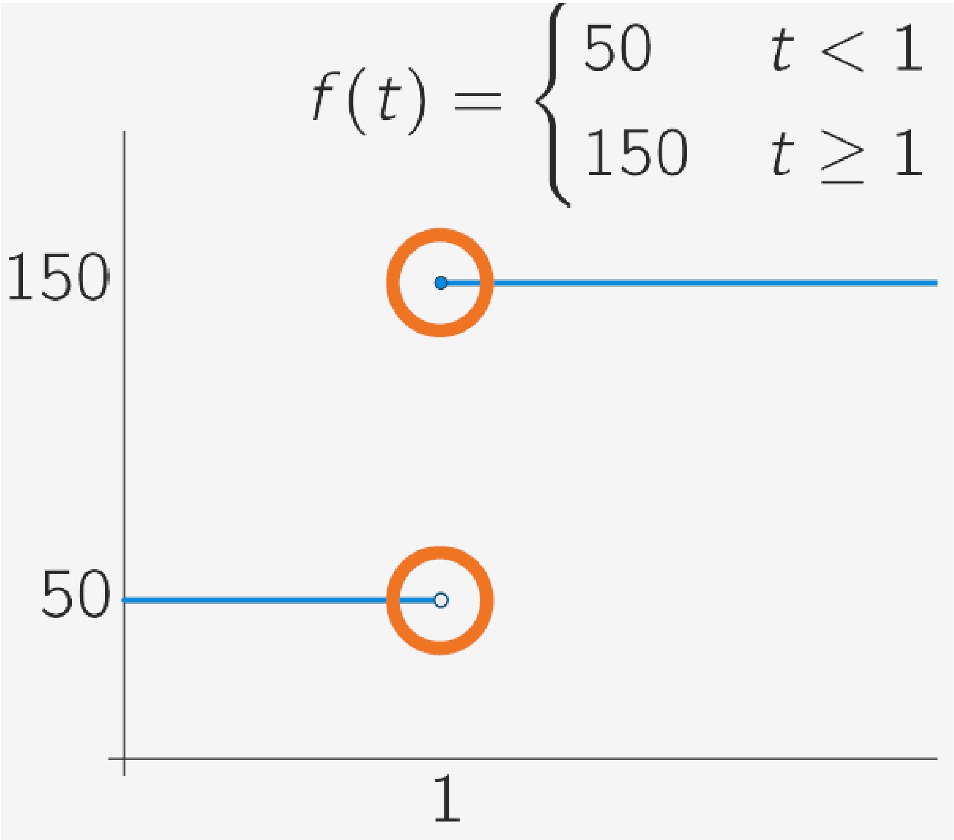
What is a continuous function? **And what is a piecewise-defined function**?

- ‘**f is continuous** at x = p: a small change on the x – axis results small changes on the y – axis’
- A function **that is continuous function in any point of its domain** is called continuous function; if a function is not continuous in a point is not included within its domain the function is still continuous
- A continuous function demonstrates **the property of the intermediate value**: if f is continuous on [a,b] then it attains all values between f(a) and f(b)

Continuity

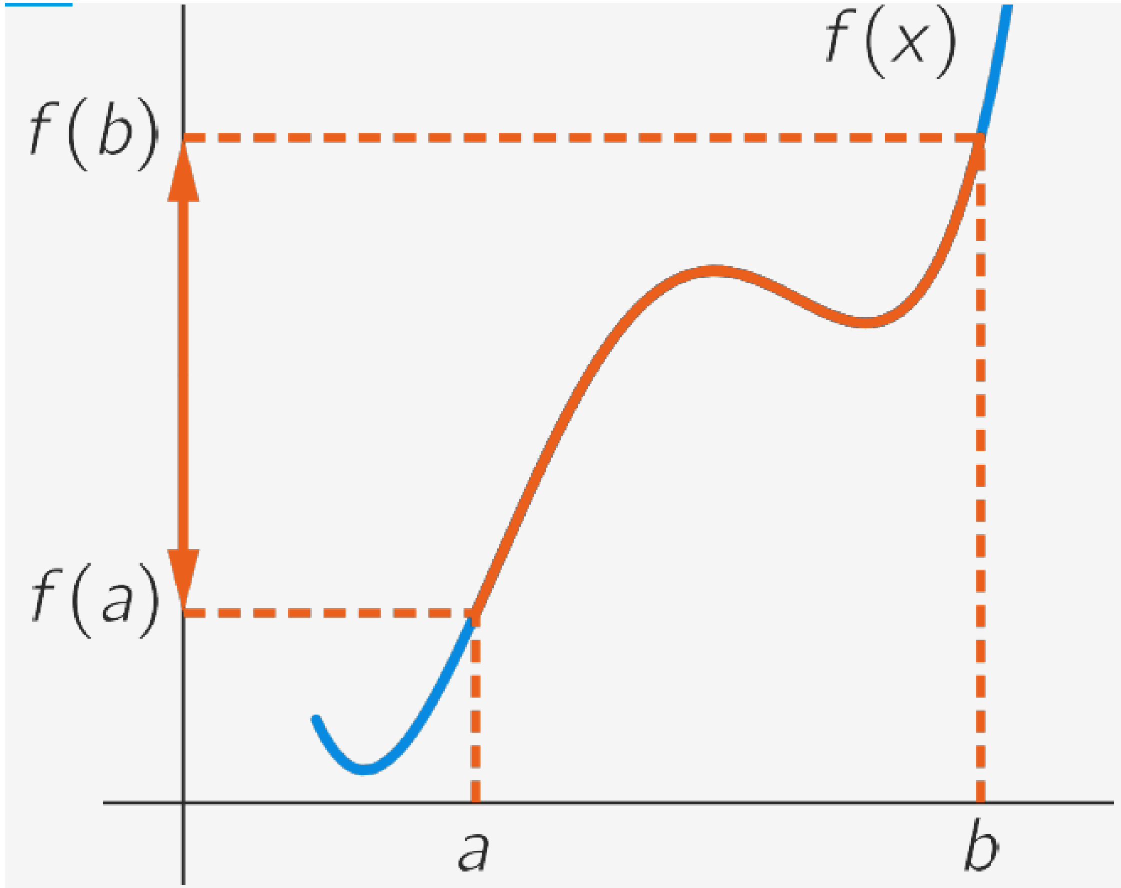


A continuous function



A noncontinuous function

Continuity

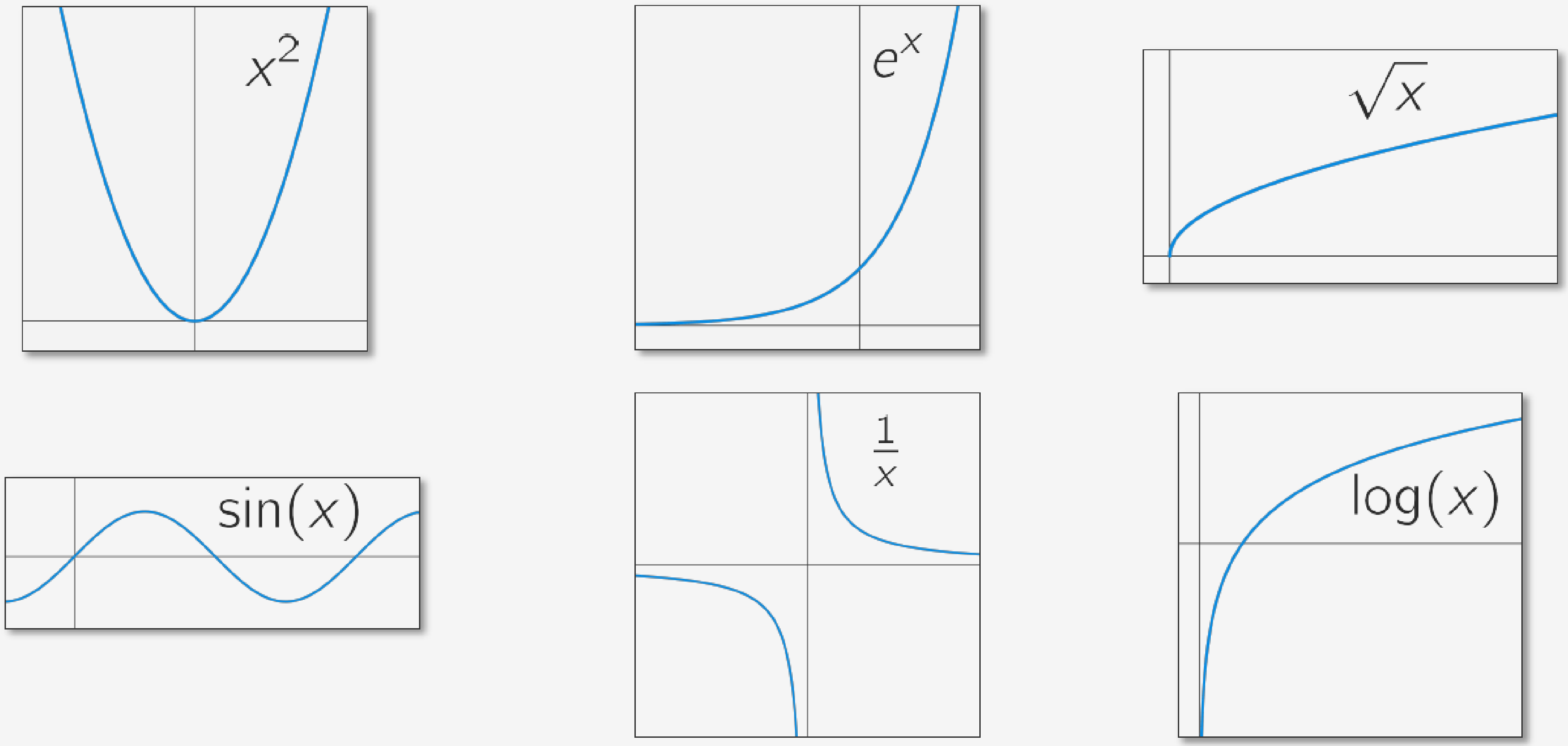


Intermediate value property

If f is continuous on $[a, b]$, then it attains all values between $f(a)$ and $f(b)$.

Continuity

Examples of continuous functions



Continuity

Combining continuous functions

If f and g are continuous, then

- $f + g$
- $f \times g$
- $\frac{f}{g}$ $\quad g(x) \neq 0$

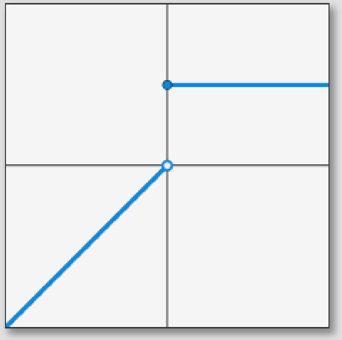
are also continuous.

$\sqrt{x}$ and $x - 1$ are continuous

↓

$\frac{\sqrt{x}}{x-1}$ for $x \geq 0, \;\; x \neq 1$

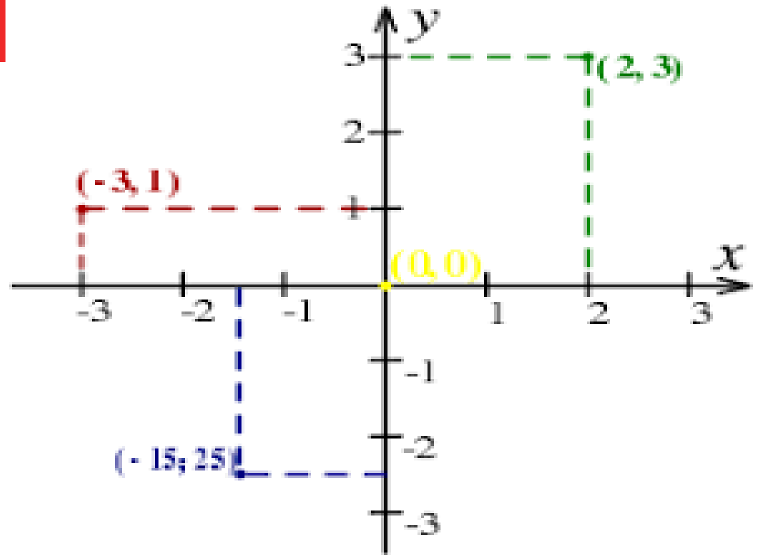
is also continuous.

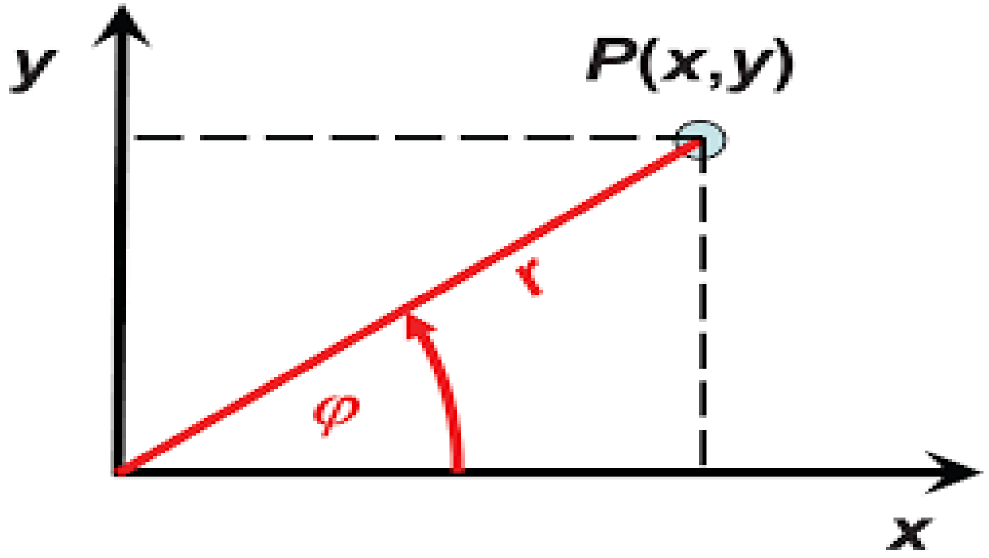


$$f(x) = \begin{cases} x & x < 0 \\ 1 & x \geq 0 \end{cases}$$ is not continuous

Coordinate systems

Coordinate systems





Cartesian coordinate system

O: the origin of the coordinate system (*0,0*). Every other point is represented with a set of numbers (*x*, *y*)

Polar coordinate system

The position of a point is defined from the set of numbers (r,θ)

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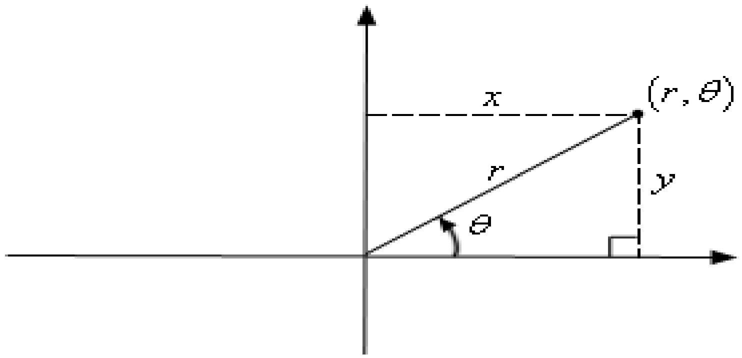
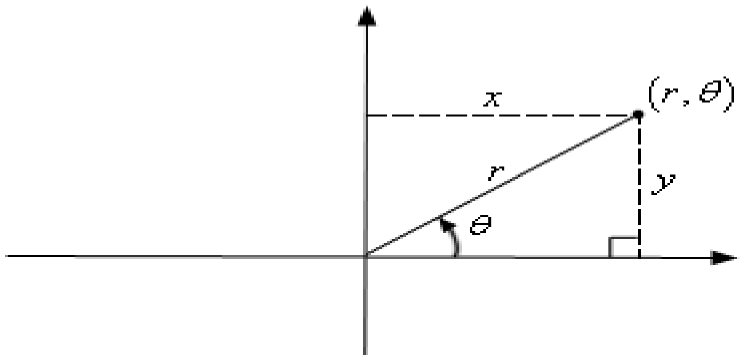
Transforming Cartesian coordinates to polar



$$\sin(\theta) = \frac{y}{r} \Rightarrow \theta = \arcsin\left(\frac{y}{r}\right)$$
$$\cos(\theta) = \frac{x}{r} \Rightarrow \theta = \arccos\left(\frac{x}{r}\right)$$
$$\tan(\theta) = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$$

$$r^2 = x^2 + y^2$$

Transforming polar coordinates to Cartesian



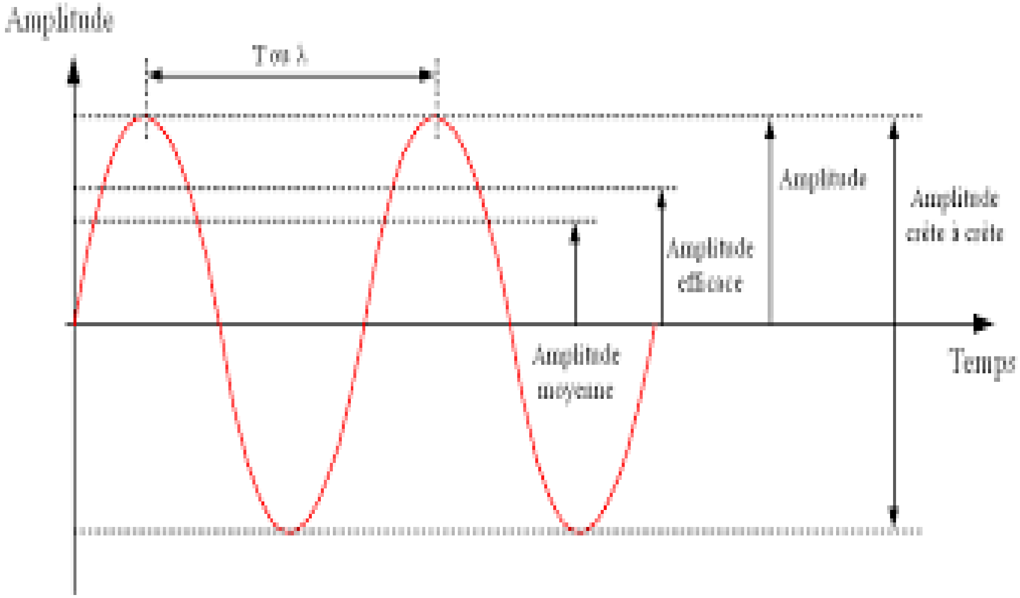
$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Functions – Part II

PART II

Basic trigonometry

Basic Trigonometry



- **Trigonometry** is important to applications in Mathematics and to foundations of calculus
- It is extremely useful in calculus, physics, engineering and the most of the sciences
- The power of the trigonometric functions is that they are periodical; its values repeat in certain intervals of domain

Basic Trigonometry

$$T^2 = \frac{4\pi^2}{GM} a^3$$

can be expressed as simply

$$T^2 = a^3$$

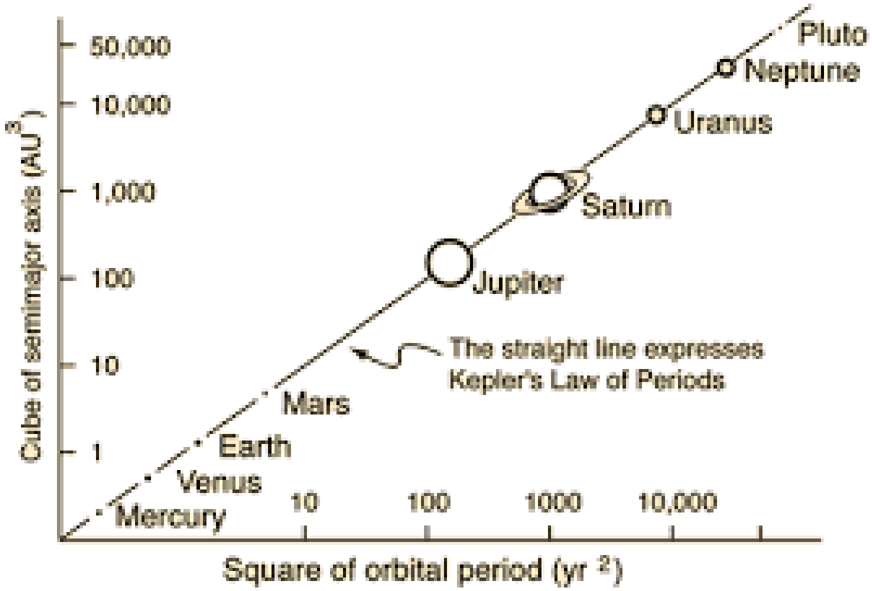
If expressed in the following units:

T Earth years

a Astronomical units AU (a = 1 AU for Earth)

M Solar masses $M_\odot$

Then $\frac{4\pi^2}{G} = 1$



Real Life Examples of Periodical Functions

- Energy Waves
- Biorhythms
- Motion of Planets

Basic Trigonometry

Basic trigonometry

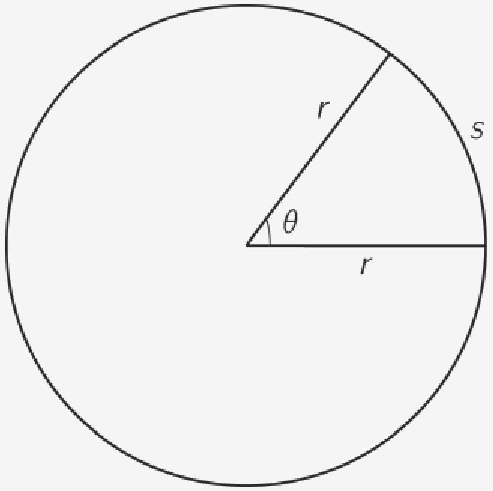
In mathematics the angles are measured in radians than in degrees

$$2\pi \text{ rad} = 360^\circ$$

$$\pi \text{ rad} = 180^\circ$$

$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ$$

$$1^\circ = \frac{\pi}{180} \text{ rad}$$



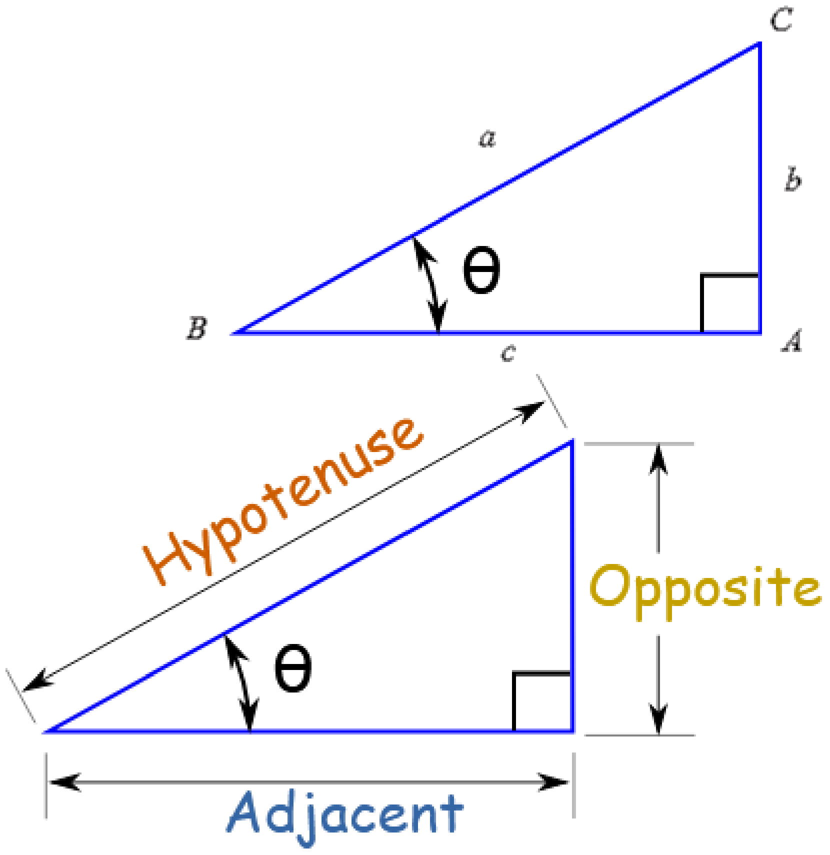
$$r = 1$$

$$\downarrow$$

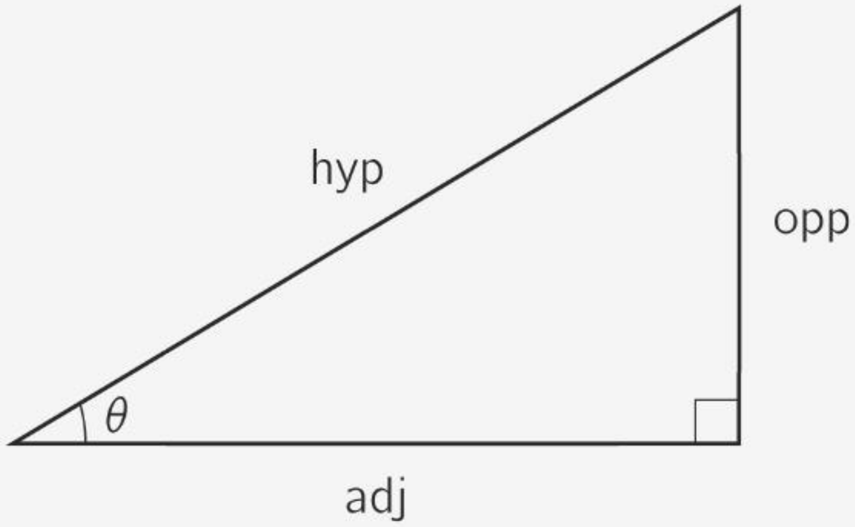
$$s = \frac{\theta}{2\pi} \cdot 2\pi\, r = r \cdot \theta$$

$$s = \theta$$

Basic trigonometry



Basic trigonometry



$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}}$$

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}}$$

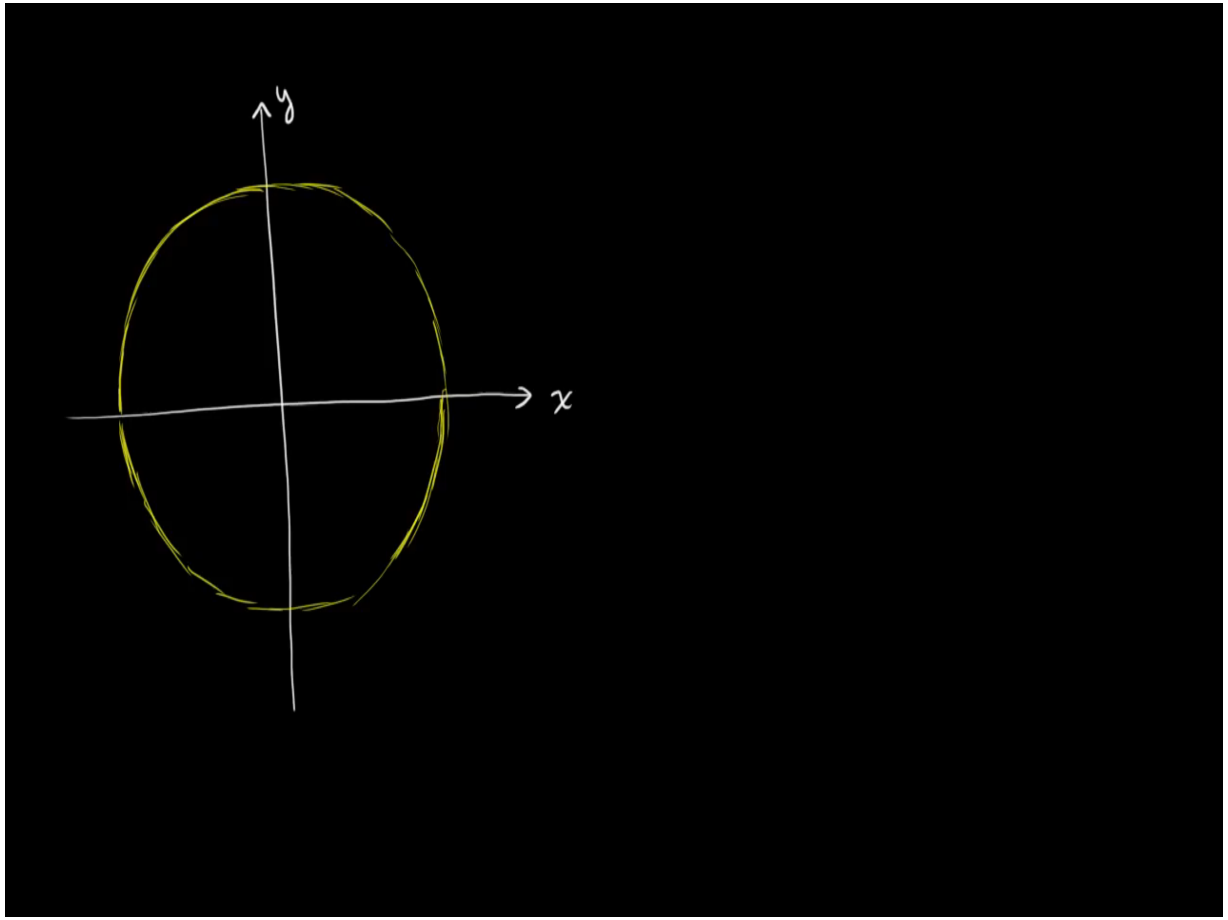
$$\tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{\sin(\theta)}{\cos(\theta)}$$

$$-1 \leq \cos(B) \leq 1$$

$$-1 \leq \sin(B) \leq 1$$

$$sin^2(B) + cos^2(B) = 1$$

Basic Trigonometry



Basic trigonometry

More…..

1. $\sin\theta = \frac{y}{r}$

2. $\csc\theta = \frac{r}{y}$

3. $\cos\theta = \frac{x}{r}$

4. $\sec\theta = \frac{r}{x}$

5. $\tan\theta = \frac{y}{x}$

6. $\cot\theta = \frac{x}{y}$

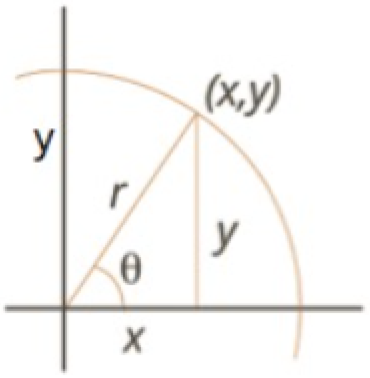


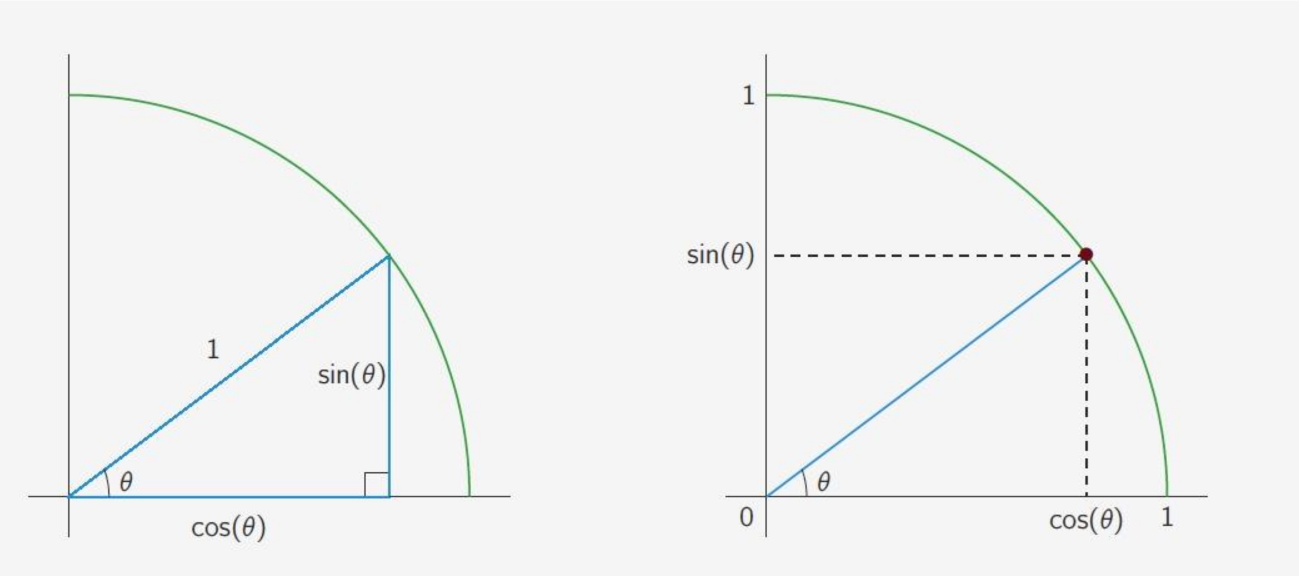
Figure 4.0-1

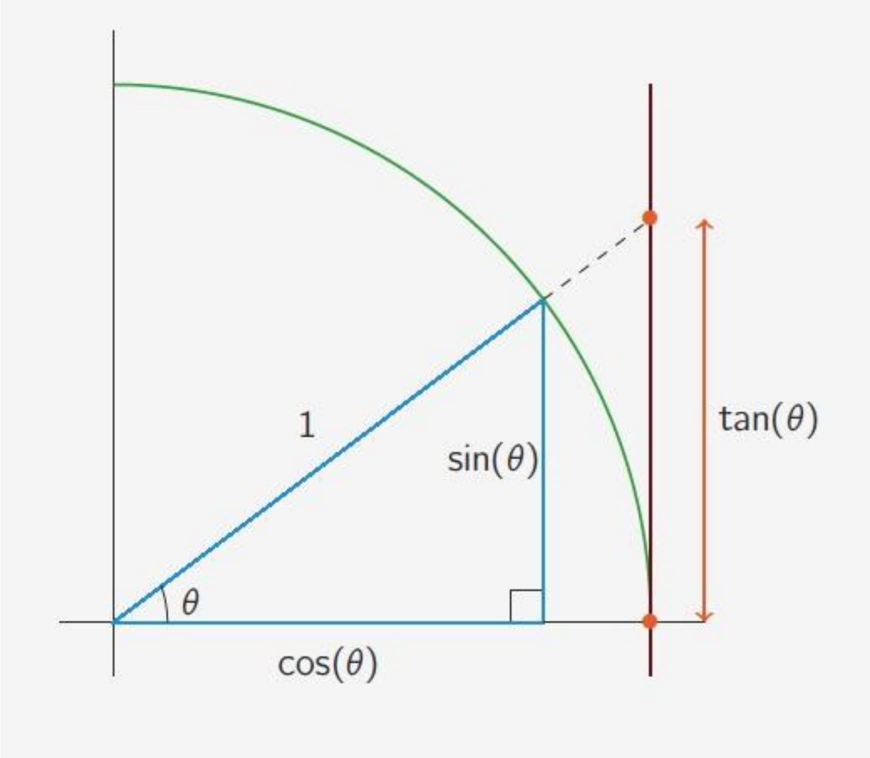
Inverse trigonometric functions

$$\text{if } x = cos\,(y) \text{ then } y = cos^{-1}(x) \text{ or } y = arcos(x)$$
$$\text{if } x = tan\,(y) \text{ then } y = tan^{-1}(x) \text{ or } y = artan(x)$$

Basic trigonometry

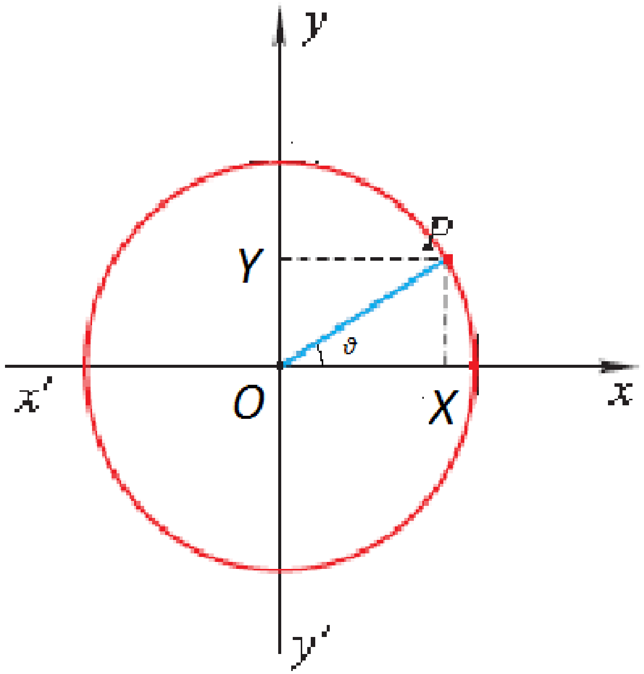
The definition of the trigonometric functions through the unit cycle

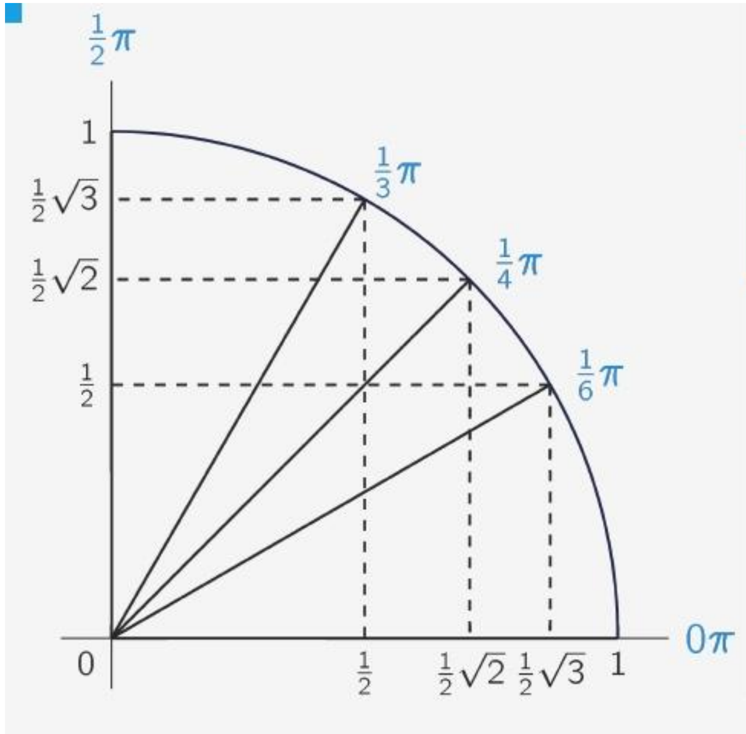




The trigonometric circle

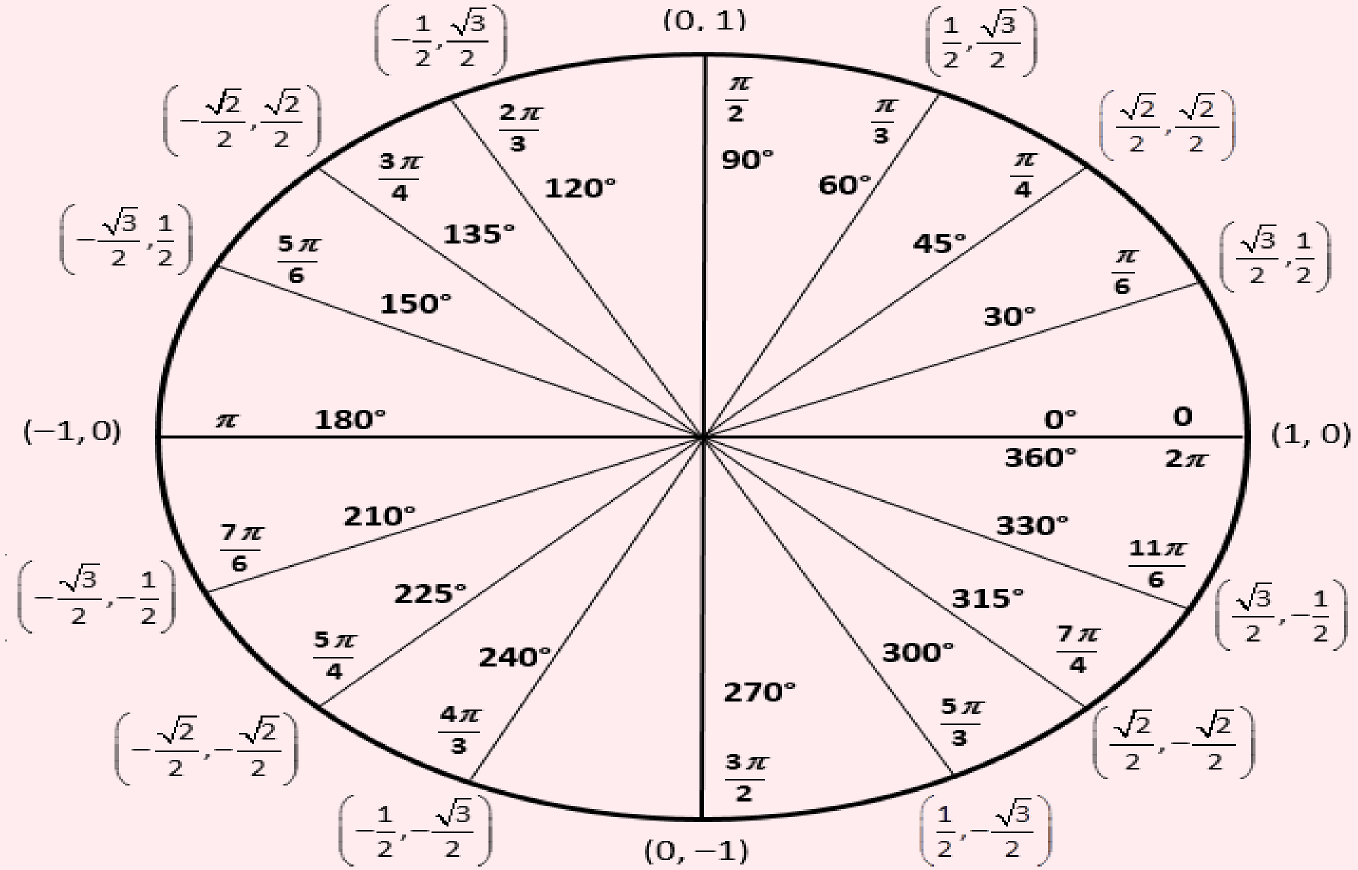
Quarter	1st	2nd	3rd	4rth
$sin(\theta)$	+	+	-	-
$cos(\theta)$	+	-	-	+





	0°	30°	45°	60°	90°
θ	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
$\cos(\theta)$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	–

The trigonometric circle



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Angle φ in radians	Angle φ in degrees	$sin\,(\varphi)$	$cos\,(\varphi)$	$\tan\,(\varphi)$
0 (=2π)	0 (=360°)	0	1	0
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90°	1	0	Not defined
$\frac{2\pi}{3}$	120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$\frac{3\pi}{4}$	135°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
$\frac{5\pi}{6}$	150°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
π	180°	0	-1	0
$\frac{7\pi}{6}$	210°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{5\pi}{4}$	225°	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
$\frac{4\pi}{3}$	240°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
$\frac{3\pi}{2}$	270°	-1	0	Not defined
$\frac{5\pi}{3}\,(=-\frac{\pi}{3})$	300 (=-60°)	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
$\frac{7\pi}{4}\,(=-\frac{\pi}{4})$	315 (=-45°)	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
$\frac{11\pi}{6}\,(=-\frac{\pi}{6})$	330 (=-30°)	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$

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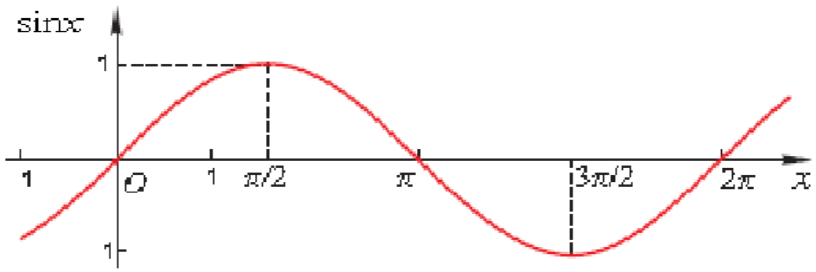
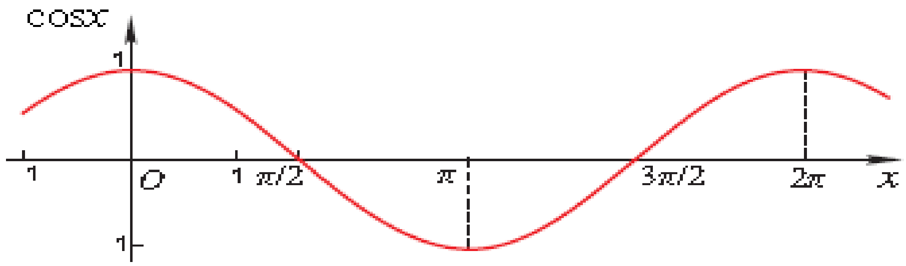
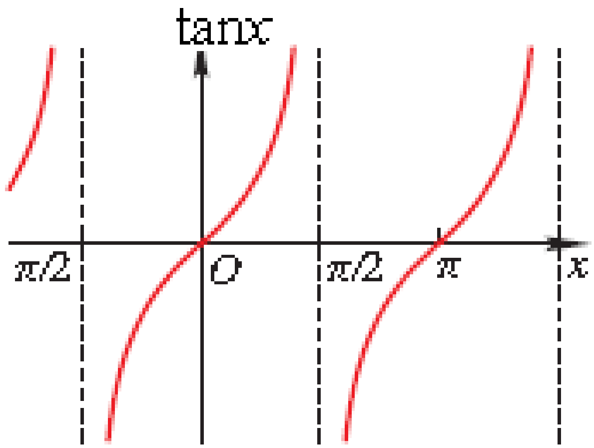
Graphs of the basic trigonometric functions







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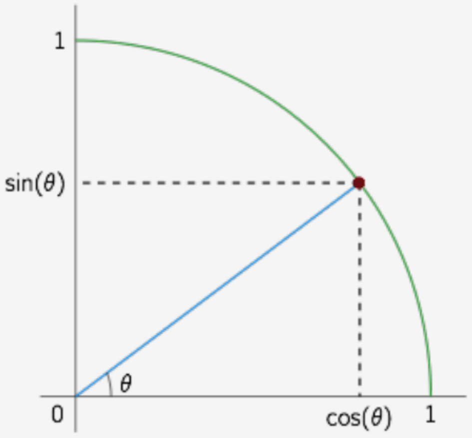
The Trigonometric Functions

What are the rules of calculation for trigonometric functions? Using the **Pythagorean Theorem** many rules can be derived. Others are more easily interpreted by considering **the graph of the trigonometric functions**, or by considering their **definition in the unit circle**

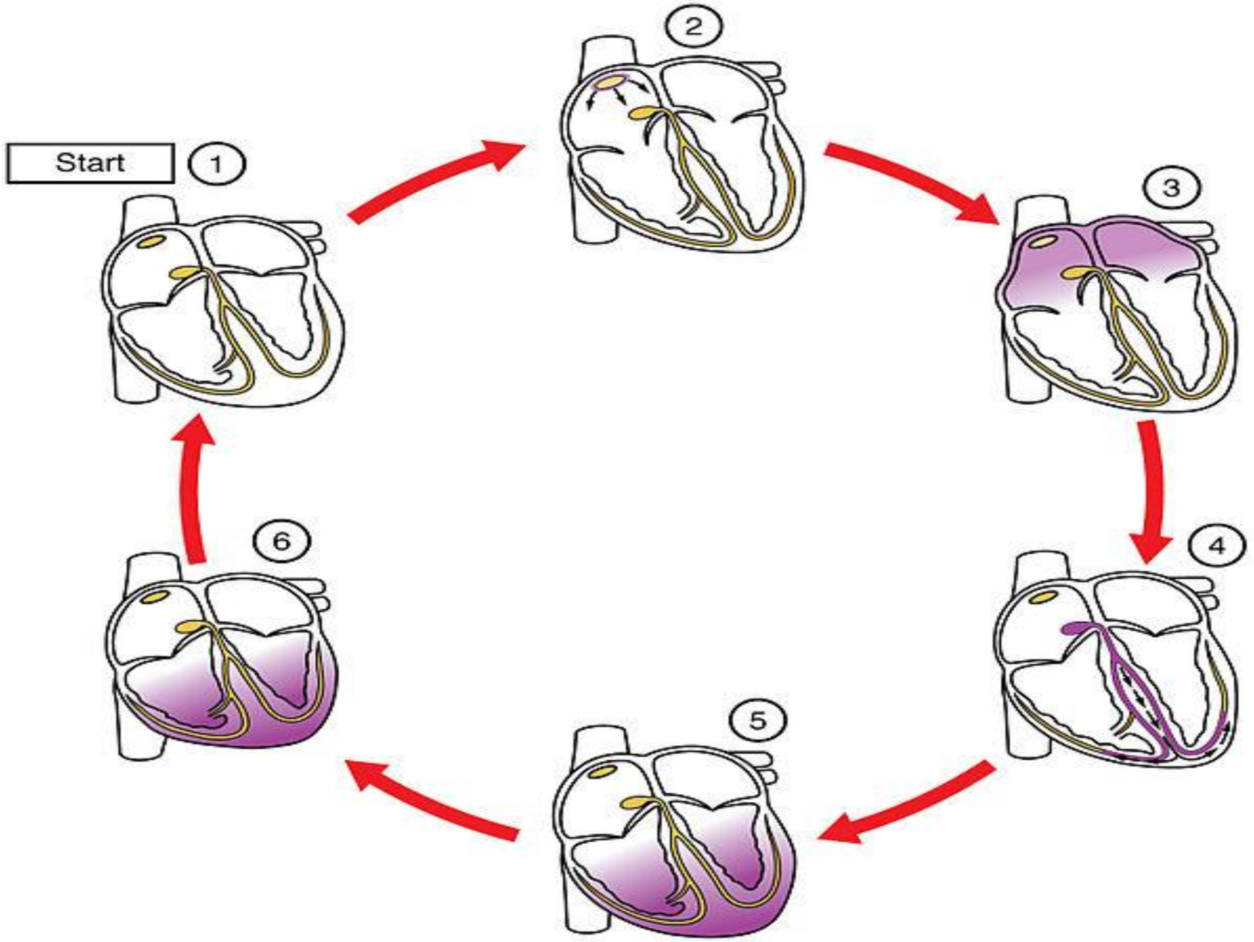
Pythagorean Theorem



Pythagorean Theorem:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

Periodic functions



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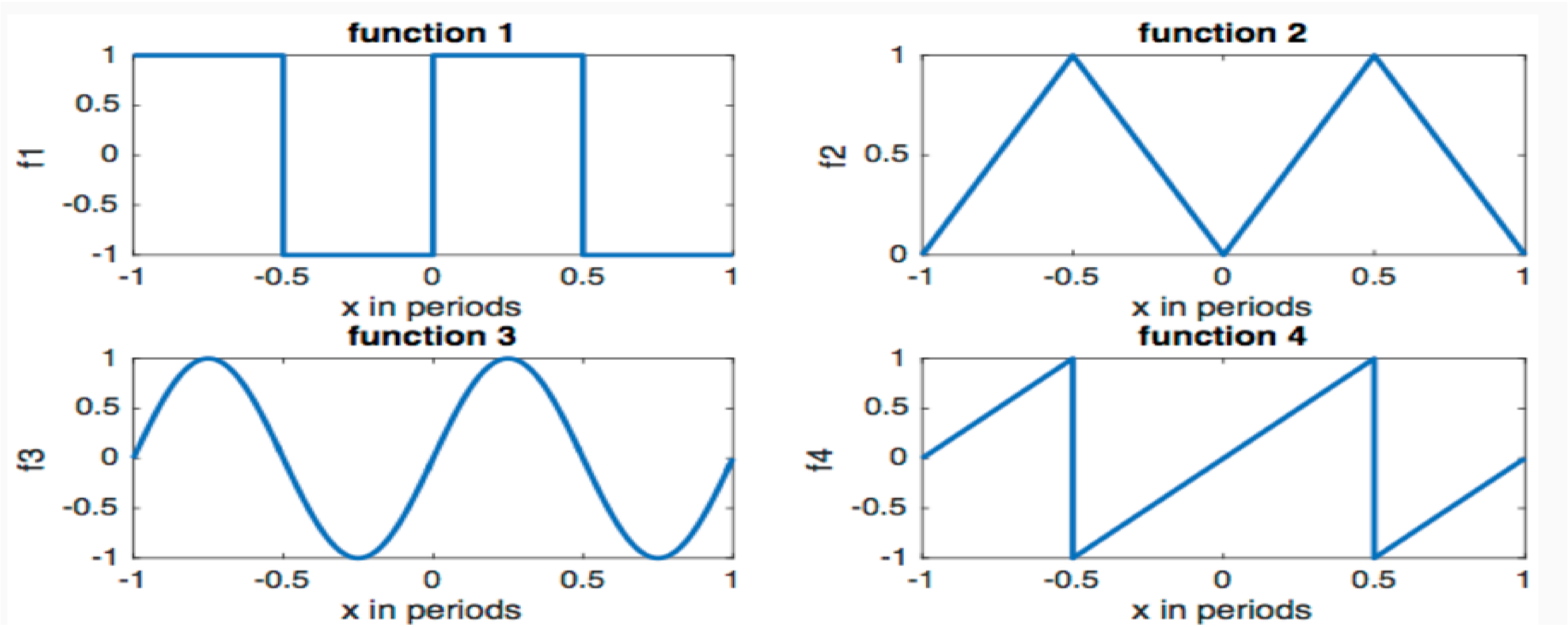
Periodic functions



Periodic functions

Definition

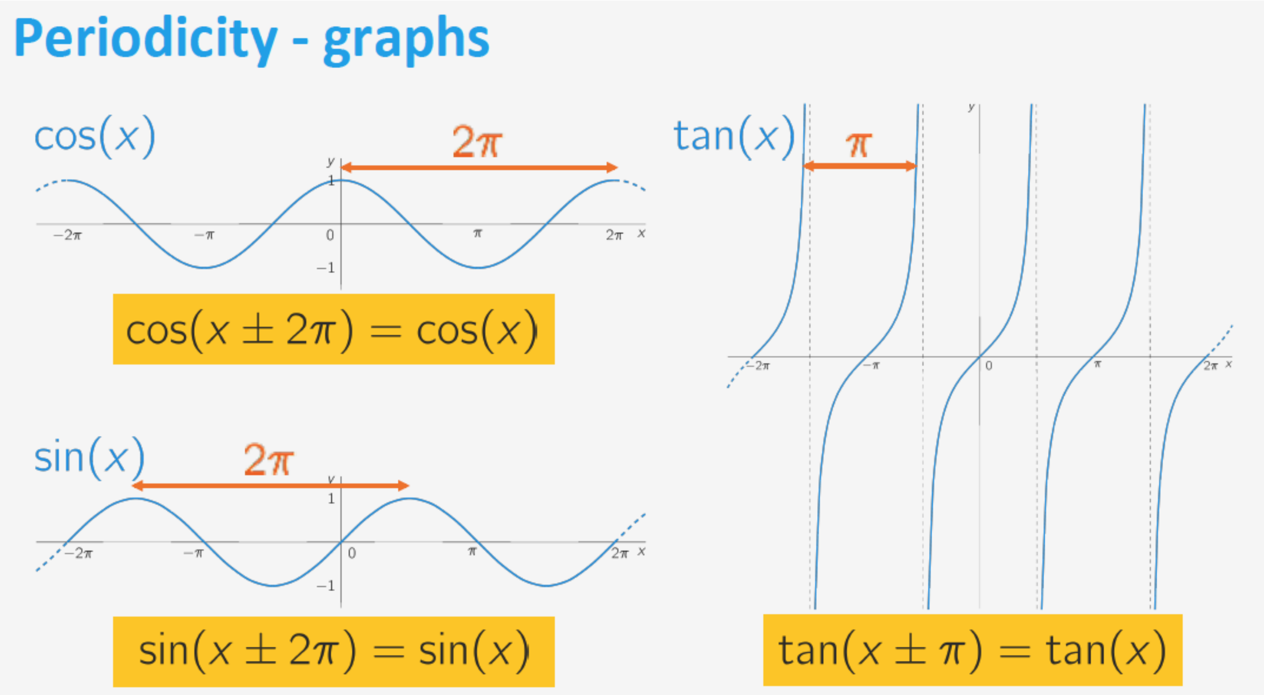
A function $f(x)$ is called periodic if, for a nonzero constant T, there is

$$f(x+T)=f(x)$$

for every x in the function's domain. The constant T is called the period of the function.

The Trigonometric Functions

- The trigonometric functions are **periodic functions. More particularly:**



$\cos(x) = \cos(x \pm 2\pi)$

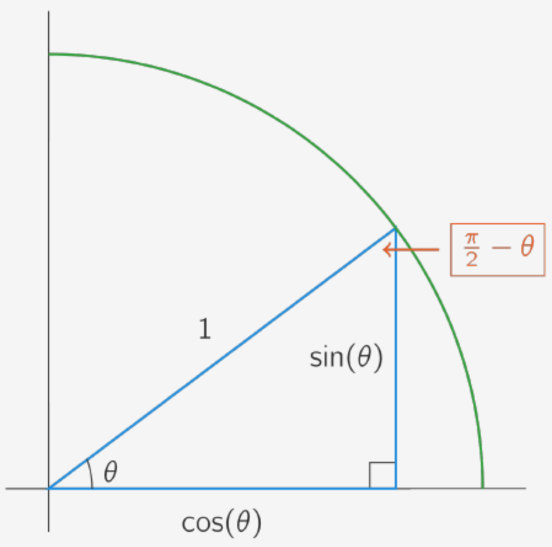
$\sin(x) = \sin(x \pm 2\pi)$

$\tan(x) = \tan(x \pm \pi)$

The Trigonometric Functions

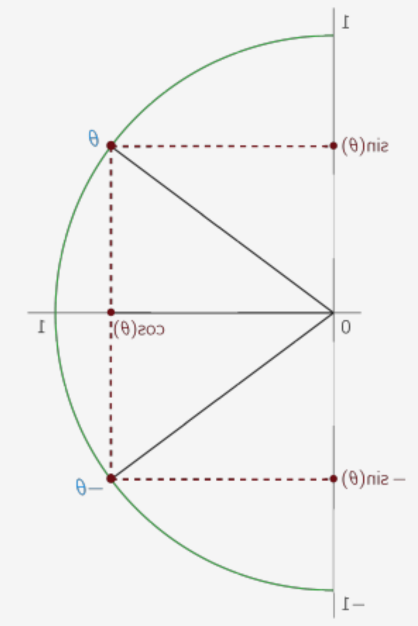
- **Rules of calculation** using trigonometric functions:

More rules of calculation



$$\cos\left(\tfrac{\pi}{2}-\theta\right)=\sin(\theta)$$

$$\sin\left(\tfrac{\pi}{2}-\theta\right)=\cos(\theta)$$



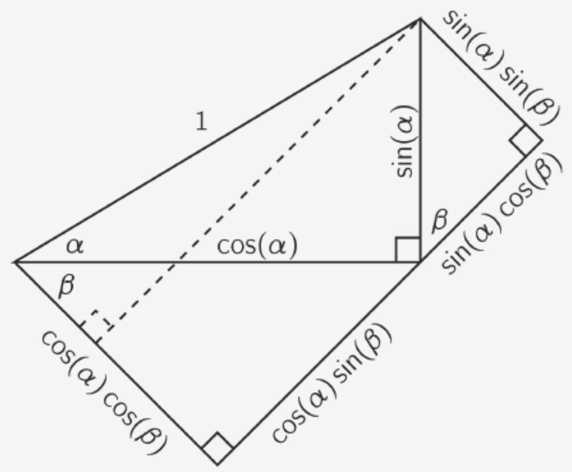
$$\cos(-\theta)=\cos(\theta)$$

$$\sin(-\theta)=-\sin(\theta)$$

The Trigonometric Functions

- **Rules of calculation** using trigonometric functions:

Double angle formulas



$$\sin(\alpha+\beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

The Trigonometric Functions

- **Rules of calculation** using trigonometric functions:

Explanation: Other trigonometric functions

In this course we only consider the trigonometric functions $\sin(x)$, $\cos(x)$ and $\tan(x)$. However, sometimes the reciprocals of these functions are given a name too (the cosecant, the secant and the cotangent respectively):

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

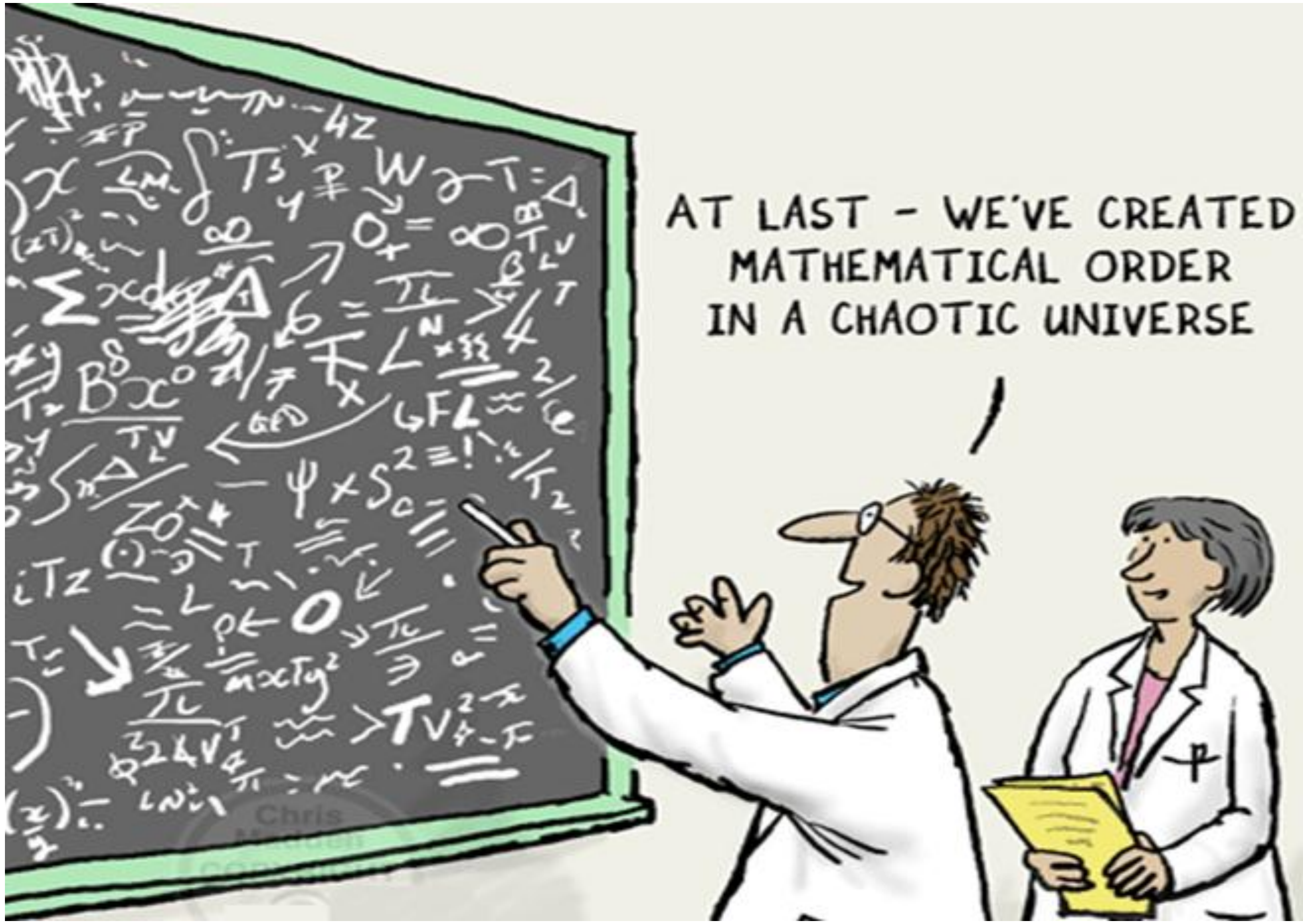
$$\cot(x) = \frac{1}{\tan(x)}$$

Useful trigonometric identities

- $sin^2(x) + cos^2(x) = 1$
- $cos^2(x) = \frac{1}{1+tan^2(x)}$
- $cos(-x) = cos(x)$
- $cos^2(x) = \frac{1+\cos(2x)}{2}$
- $\sin^2(x) = \frac{1-\cos(2x)}{2}$

- $sin(-x) = -sin(x)$
- $tan(-x) = -tan(x)$
- $sin\left(\frac{\pi}{2} \pm x\right) = cos\ (x)$
- $cos\left(\frac{\pi}{2} \pm x\right) = \mp sin\ (x)$
- $sin(\pi \pm x) = \mp sin(x)$
- $cos(\pi \pm x) = -\cos(x)$
- $tan(\pi \pm x) = \pm tan(x)$
- $sin(2\pi \pm x) = \pm sin(x)$
- $cos(2\pi \pm x) = cos(x)$
- $tan(2\pi \pm x) = \pm tan(x)$

- $sin(x \pm y) = sin(x)\,cos(y) \pm cos(x)\,sin\ (y)$
- $cos(x \pm y) = cos(x)\,cos\ (y) \mp sin(x)\,sin\ (y)$


AT LAST - WE'VE CREATED MATHEMATICAL ORDER IN A CHAOTIC UNIVERSE

Functions – Compositions

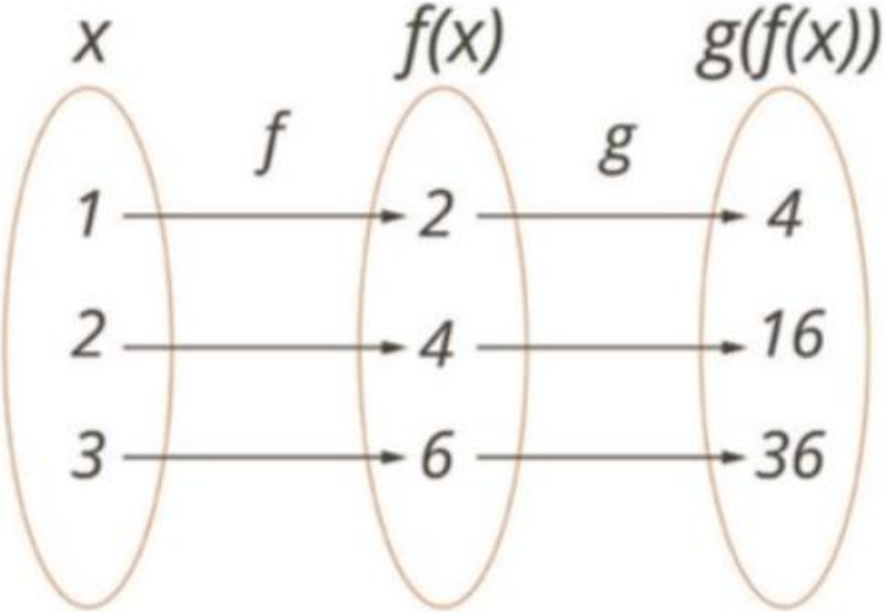
Compositions: Functions can be composed by applying a function to another function

e.g. **cos(3x+1) or exp(sqrt(x)) or ln(cos(x))**

Real life problems in the majority of the cases and in order to be described (see the work-function of an amplifier) need this type of functions (the composite ones). In this lecture we will learn **how to make these expressions simpler** !!

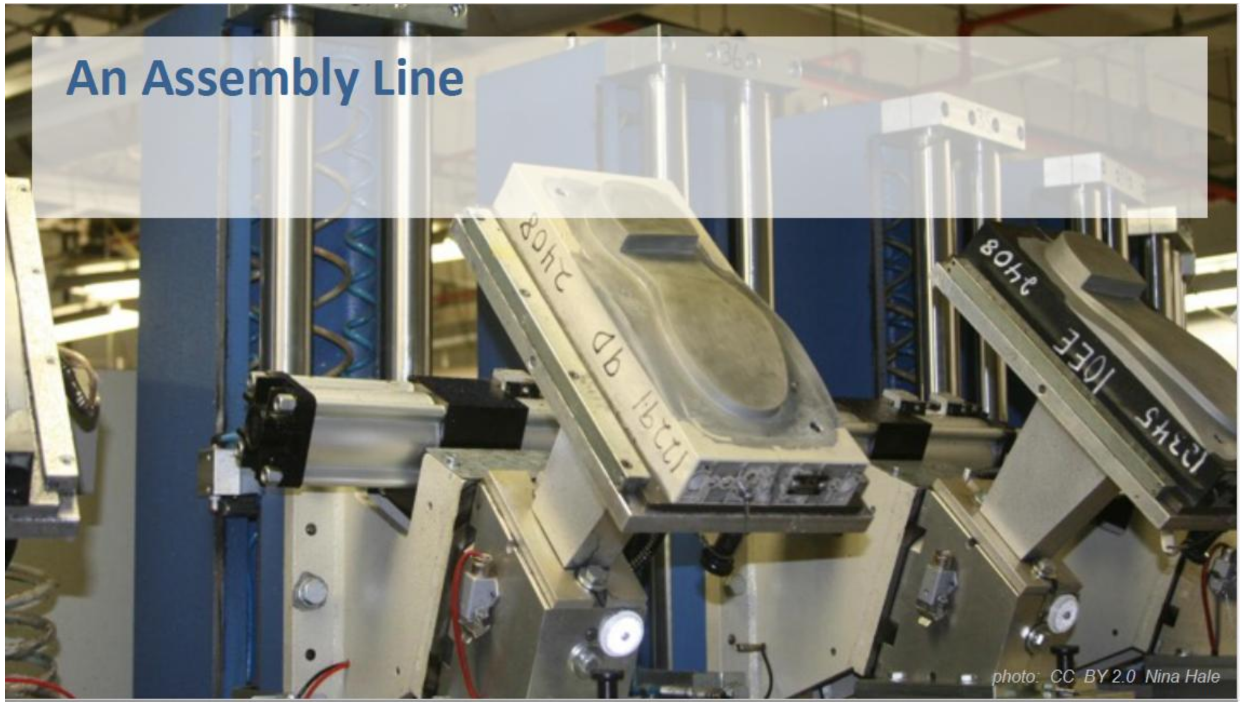
Functions - Compositions

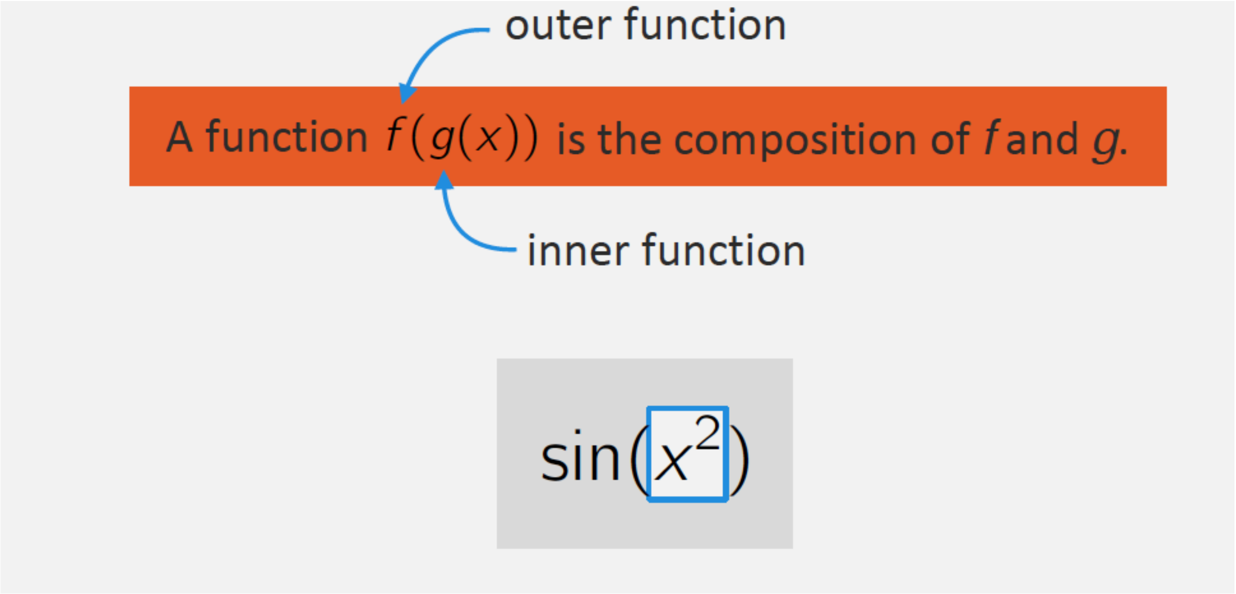
The *composition* of functions involves taking the output values from one function and using those as the input values for a second function. Visually, it looks like this:



Functions – Compositions

An example of a composite function is an Assembly Line





Functions – Compositions

- During the composition of functions **the ordering the functions operate has a significant role / impact to the final result**

- Check it for example for the functions: sin(x) and x^2 ; try to operate them in the two different orders and check your results

Functions – Compositions

- An example **how useful concept** is the composition of the functions <u>in dealing complicated functions</u> follows:

$$f(x) = x^2 + 2x + 1$$
$$g(x) = x^4$$

$$(x^4)^2 + 2x^4 + 1 = 4$$
$$p^2 + 2p + 1 = 4$$
$$p^2 + 2p - 3 = 0$$
$$(p+3)(p-1) = 0$$
$$p = -3 \quad \text{or} \quad p = 1$$
$$x^4 = -3 \quad \text{or} \quad x^4 = 1$$
$$x = 1 \quad \text{or} \quad x = -1$$

Functions – Rules of Calculation

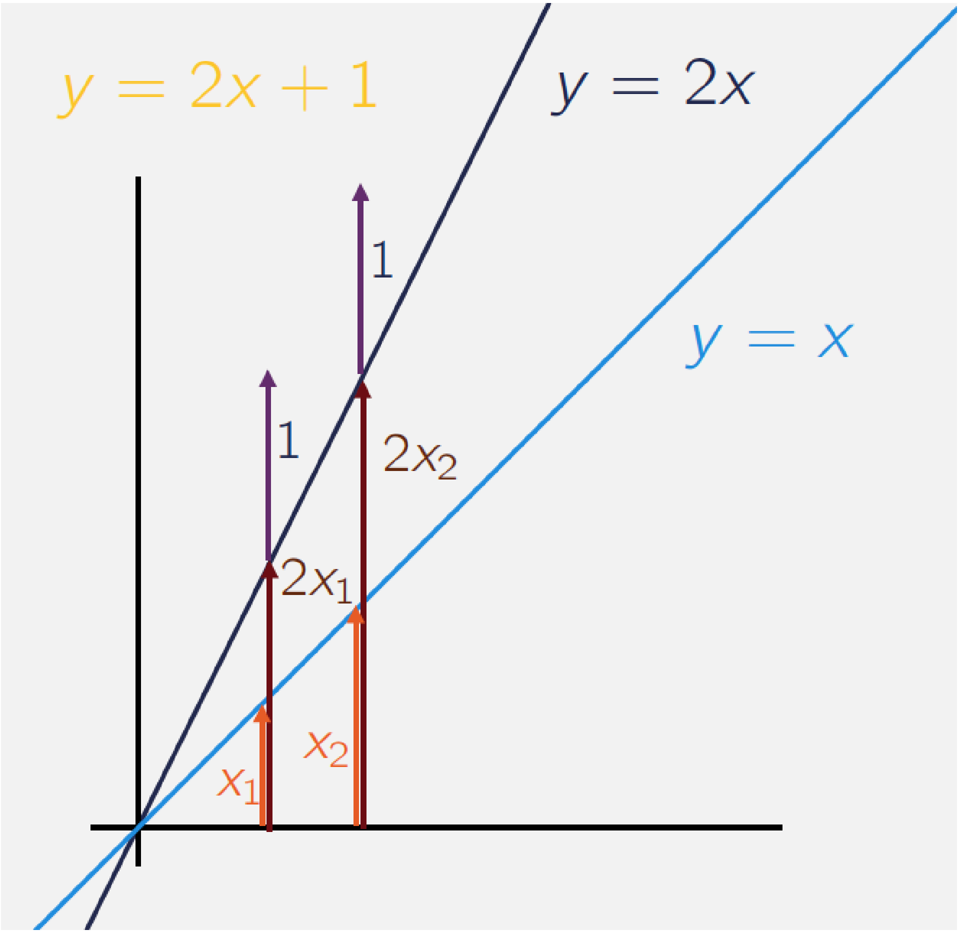
- But how does a graph of a **composite function** look like? Since we can make a composite function, combining any two functions, is impossible to be able to present all of them. We will **restrict in case we wish to make a composite with a linear function**

Functions – Rules of Calculation

The two characteristics of a linear function are: (a) **scaling**; and (b) **translation**

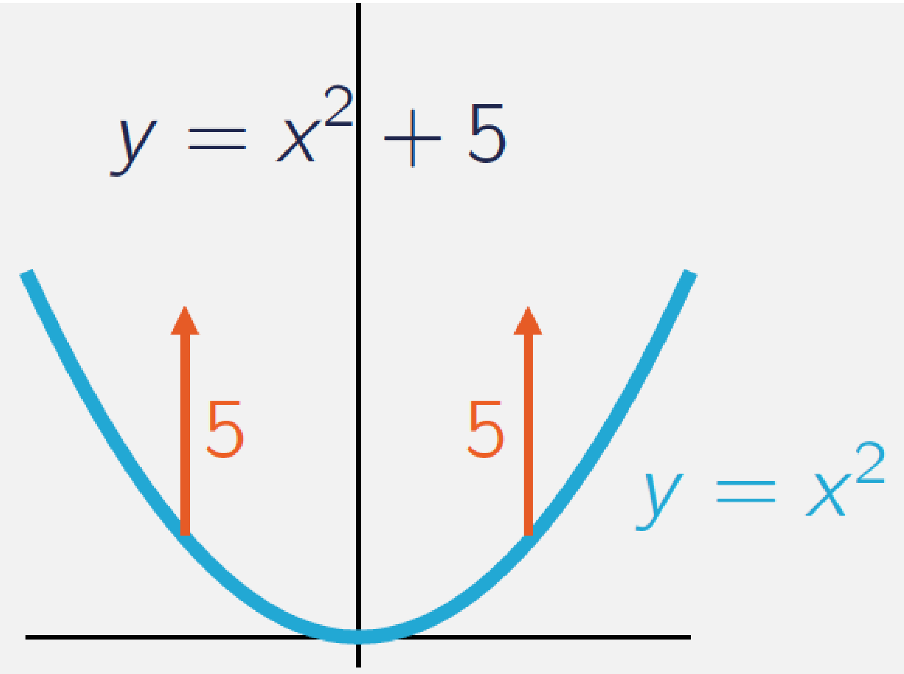
$$f(x) = 2x + 1$$

- Scaling $\times 2$
- Translation $+1$



Functions – Rules of Calculation

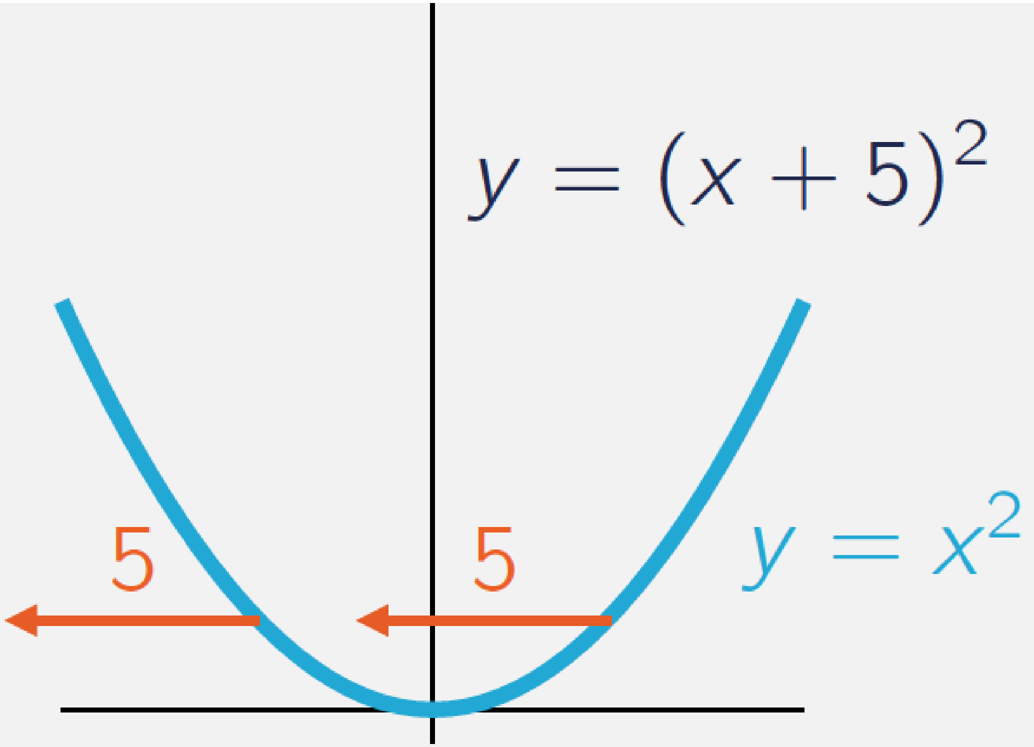
As you will see, the composition depends on the order of the action; in the below case **our function is elevated by five**



$$f(x) = x + 5$$
$$g(x) = x^2$$
$$f(g(x)) = x^2 + 5$$

Functions – Rules of Calculation

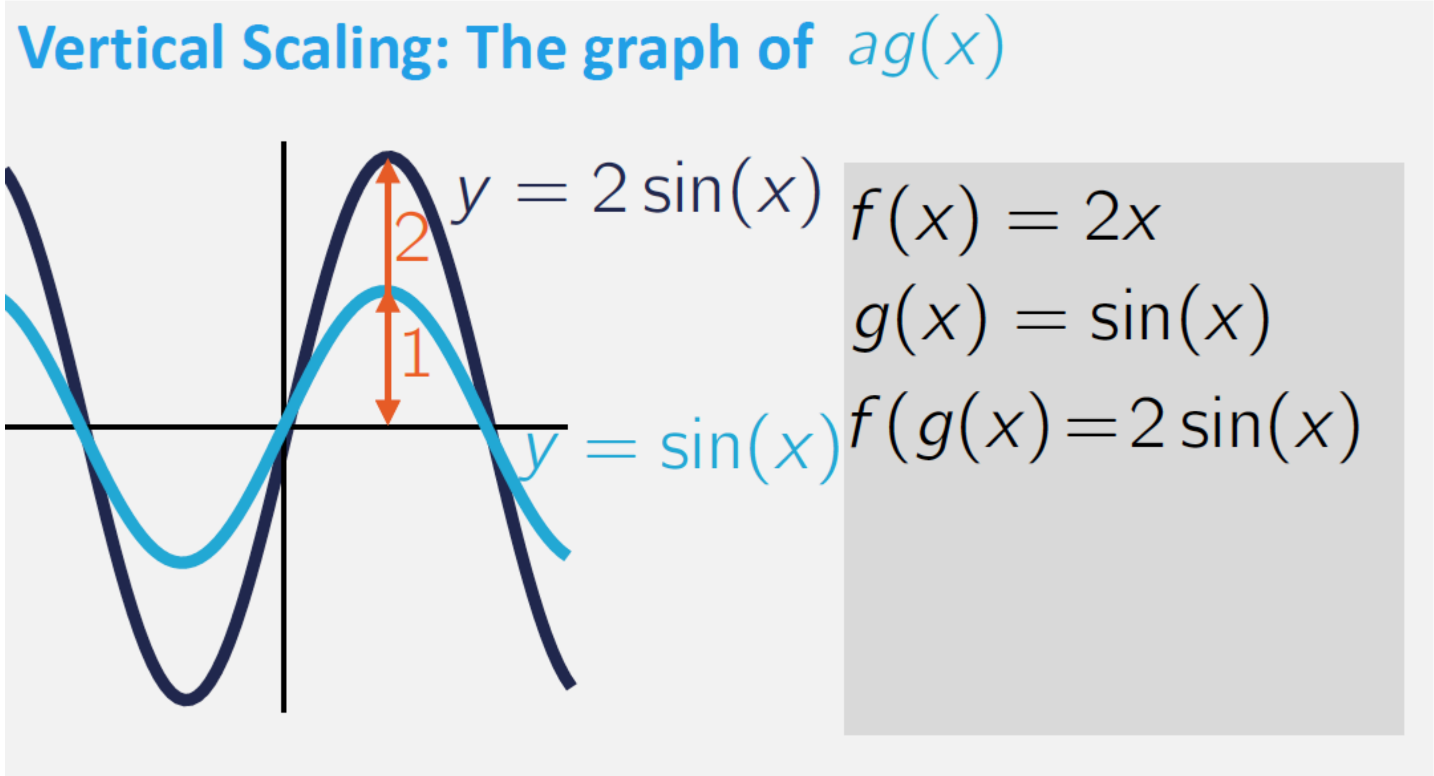
As you will see, the composition depends on the order of the action; in the below case **our function is translated towards left by five**



$$f(x) = x + 5$$
$$g(x) = x^2$$
$$g(f(x)) = (x+5)^2$$

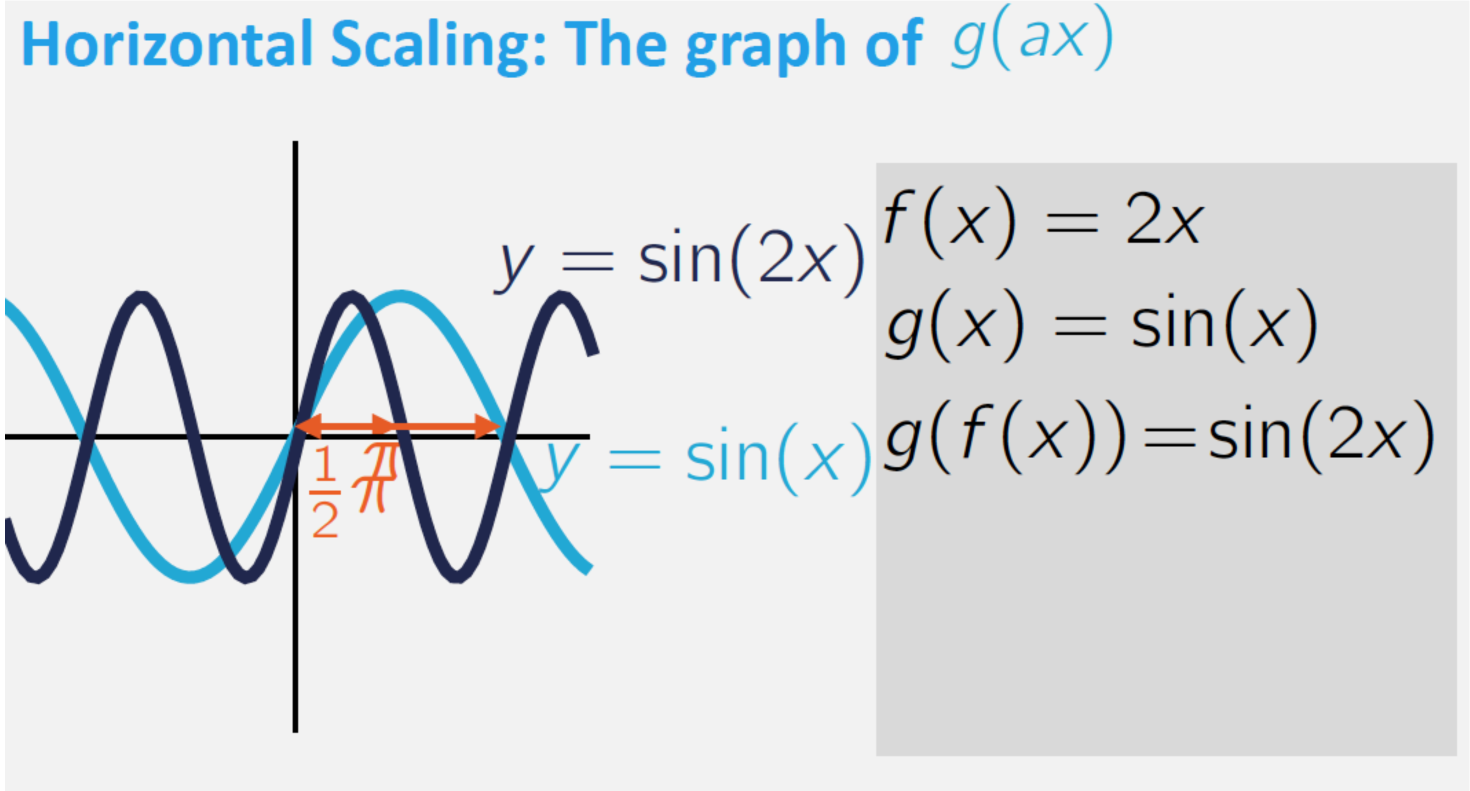
Functions – Rules of Calculation

Another effect is **scaling** as an effect of the composition of a linear function with our one; an example follows:



Functions – Rules of Calculation

In case the functions are composed in the reverse order than in the previous case of vertical scaling then the effect **has an impact to the period of the periodic function (horizontal scalling)** in this case:



Functions – Rules of Calculation

As general rule we have:

Composing with linear functions

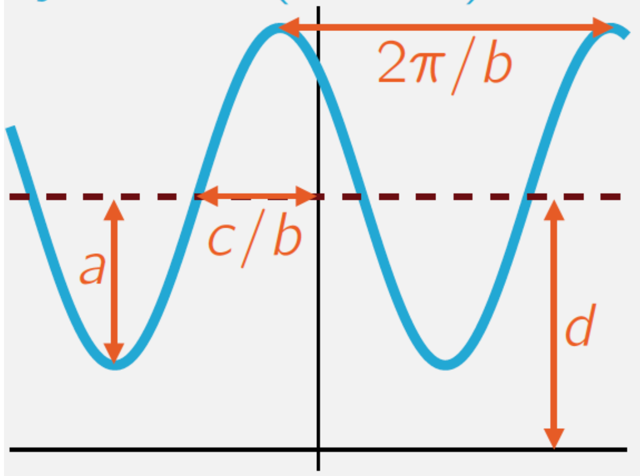
$g(x) + a$	Vertical shift upwards
$g(x + a)$	Horizontal shift to the left
$ag(x)$	Vertical scaling by a
$g(ax)$	Horizontal scaling by $1/a$

Functions – Rules of Calculation

We will apply the rules of scaling and translation to the sin(x) function. We have selected sin(x) since such functions represent a lot of physical phenomena e.g. **an electromagnetic wave, an ac current** etc

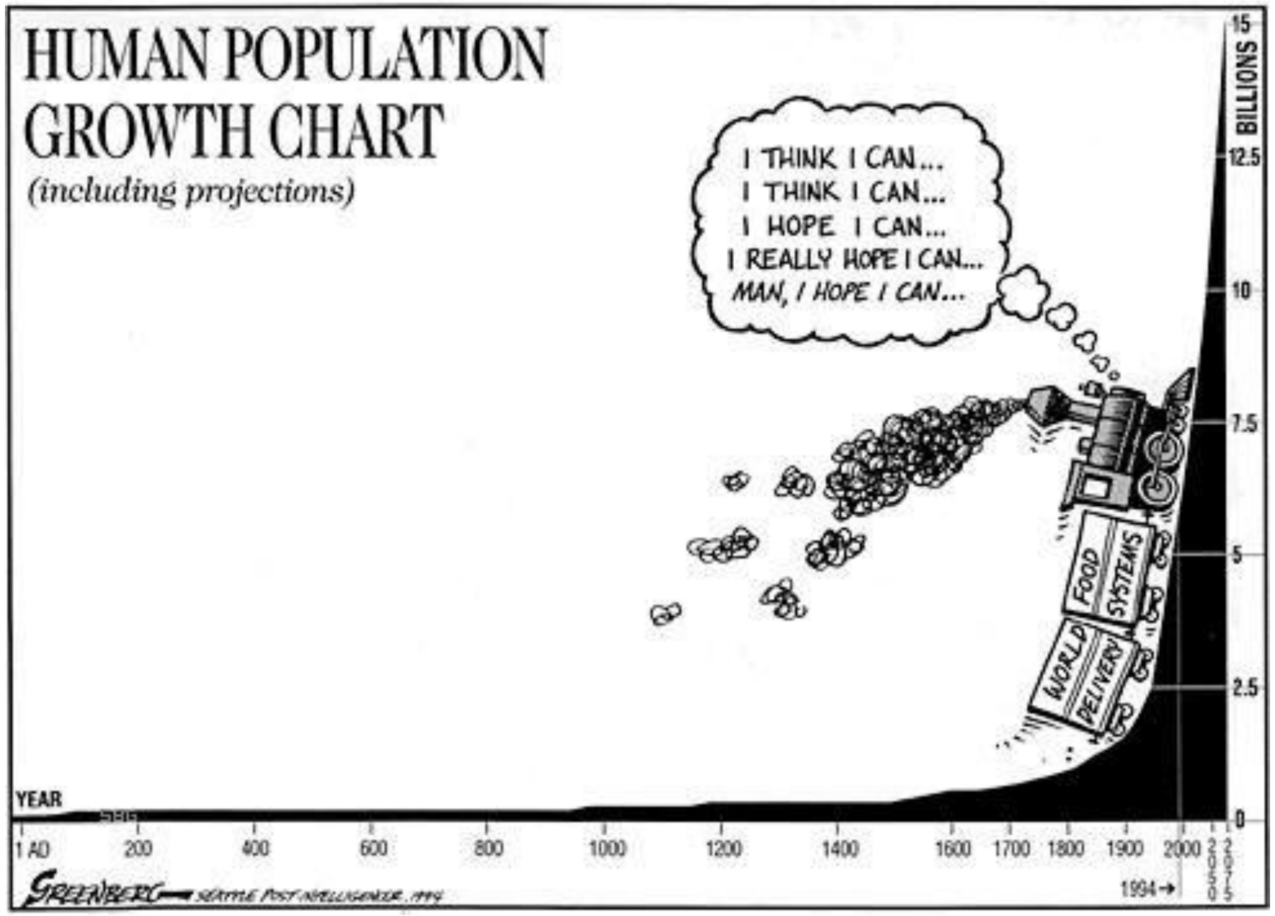
A wave function

$$y = a\sin(bt + c) + d$$



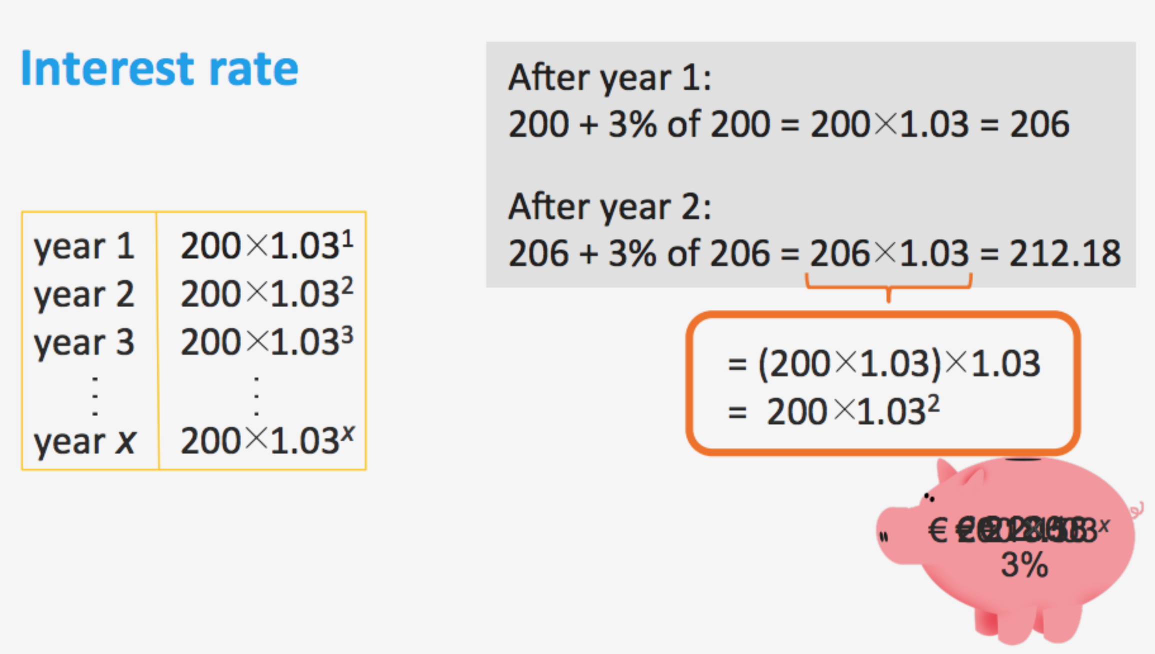
a	**Amplitude**
$2\pi/b$	**Frequency**
c	**Phase**
d	**Equilibrium height**

The Exponential Function



The Exponential Function

- The interest rate you receive in your deposits in the bank **is related to exponential functions**



The general form of an exponential function is: Ab^x where $A \neq 0$, $b>0$ and $b \neq 1$ (in order do not have the constant function)

The Exponential Function

Exponential functions

Form of an exponential function:

$$A b^x$$

- b : base
- x : exponent

$A \neq 0$

$b > 0, \; b \neq 1$

Example: 200×1.03^x

A b

The Exponential Function

What is b^x if x is not rational?

Approximation of x with rational numbers

↓

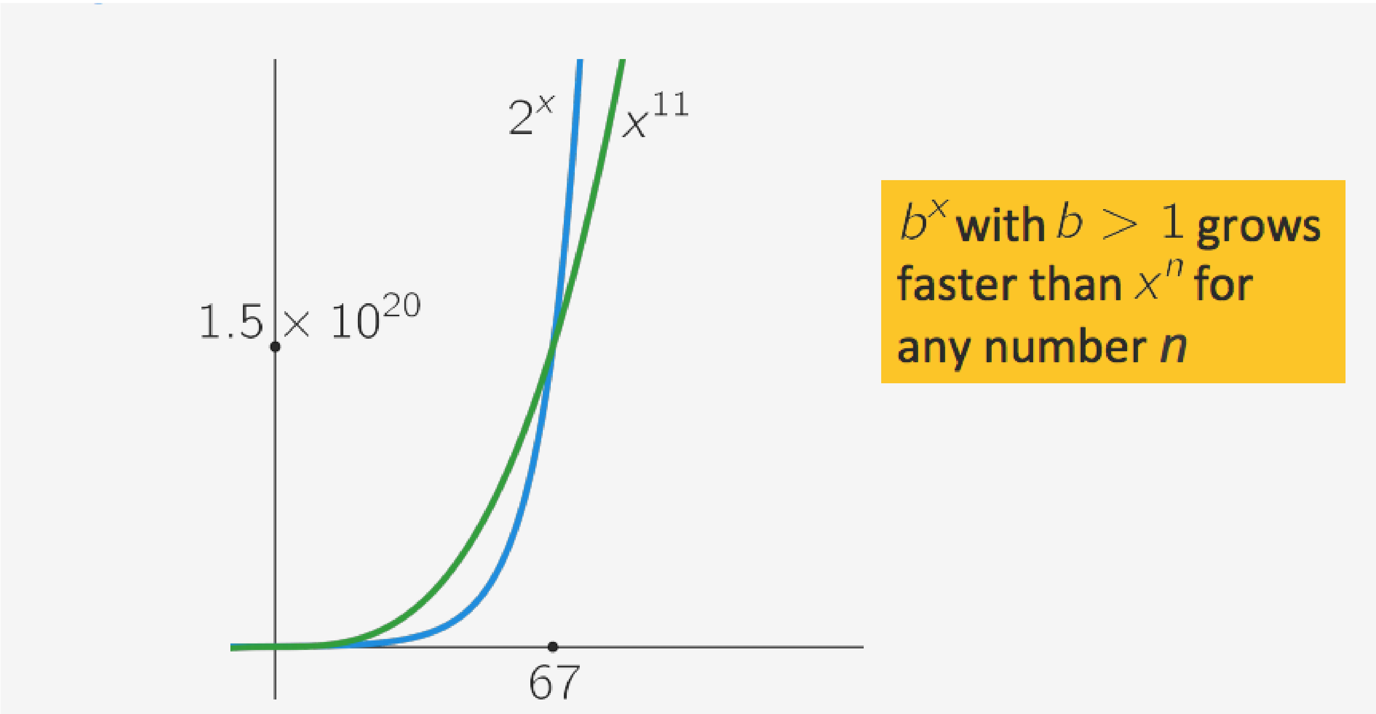
Approximation of b^x

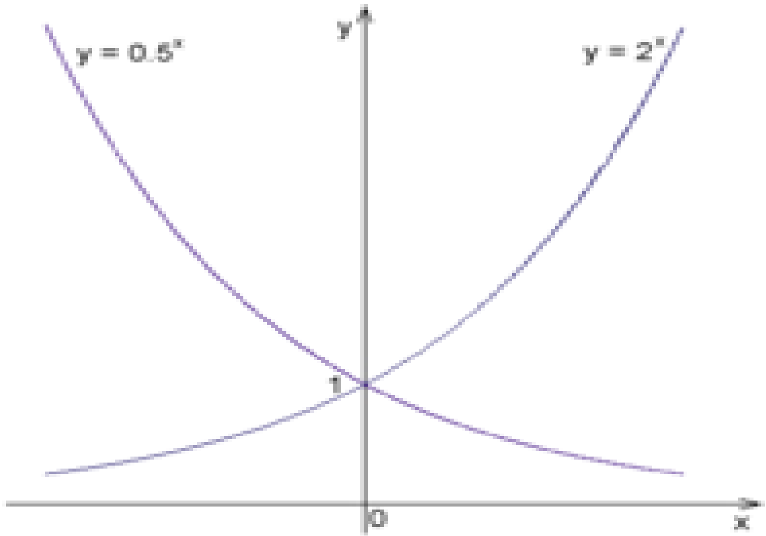
$2^{\sqrt{2}} = ?$

1.4	$2^{1.4} = 2^{\frac{14}{10}} = 2.6390\ldots$
1.41	$2^{1.41} = 2^{\frac{141}{100}} = 2.6572\ldots$
1.414	$2^{1.414} = 2^{\frac{1414}{1000}} = 2.6647\ldots$
1.4142	$2^{1.4142} = 2^{\frac{14142}{10000}} = 2.6651\ldots$
⋮	⋮
$\sqrt{2}$	$2^{\sqrt{2}} = 2.6651\ldots$

The Exponential Function

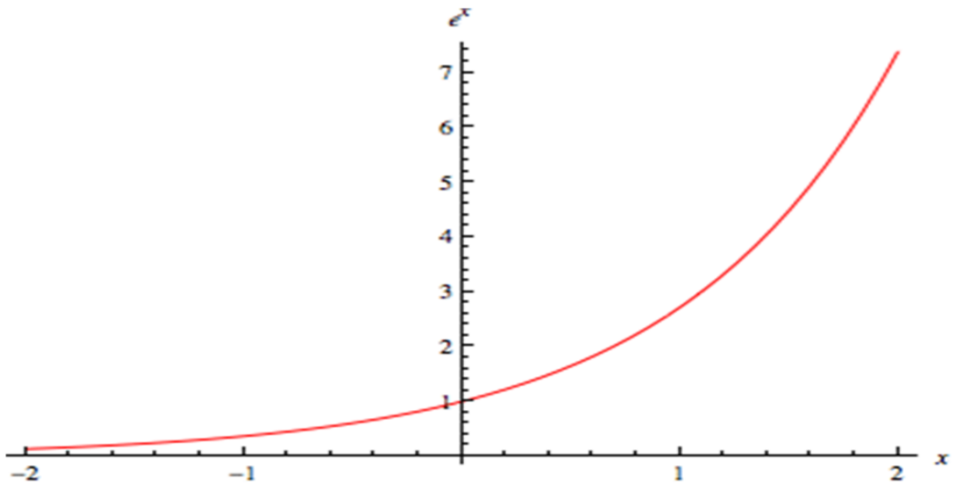
- The **exponential functions** are growing faster than any polynomial!!!





The exponential function

$$f(x) = e^x$$



- Number e is often used as base
- $e = 2.71828182845904523536028747135 27\ldots$
- e is the unique number such that

$$\frac{d}{dx}e^x = e^x$$

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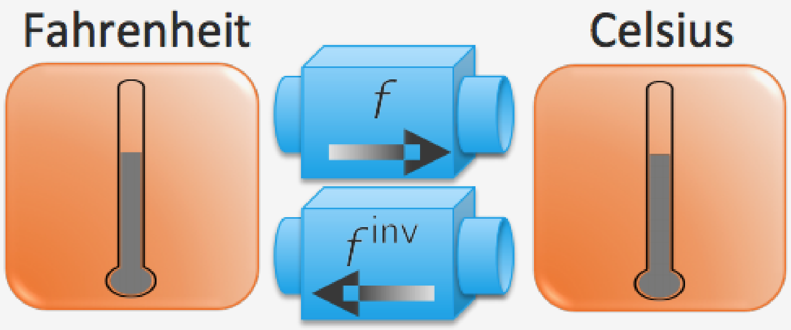
The Inverse Functions

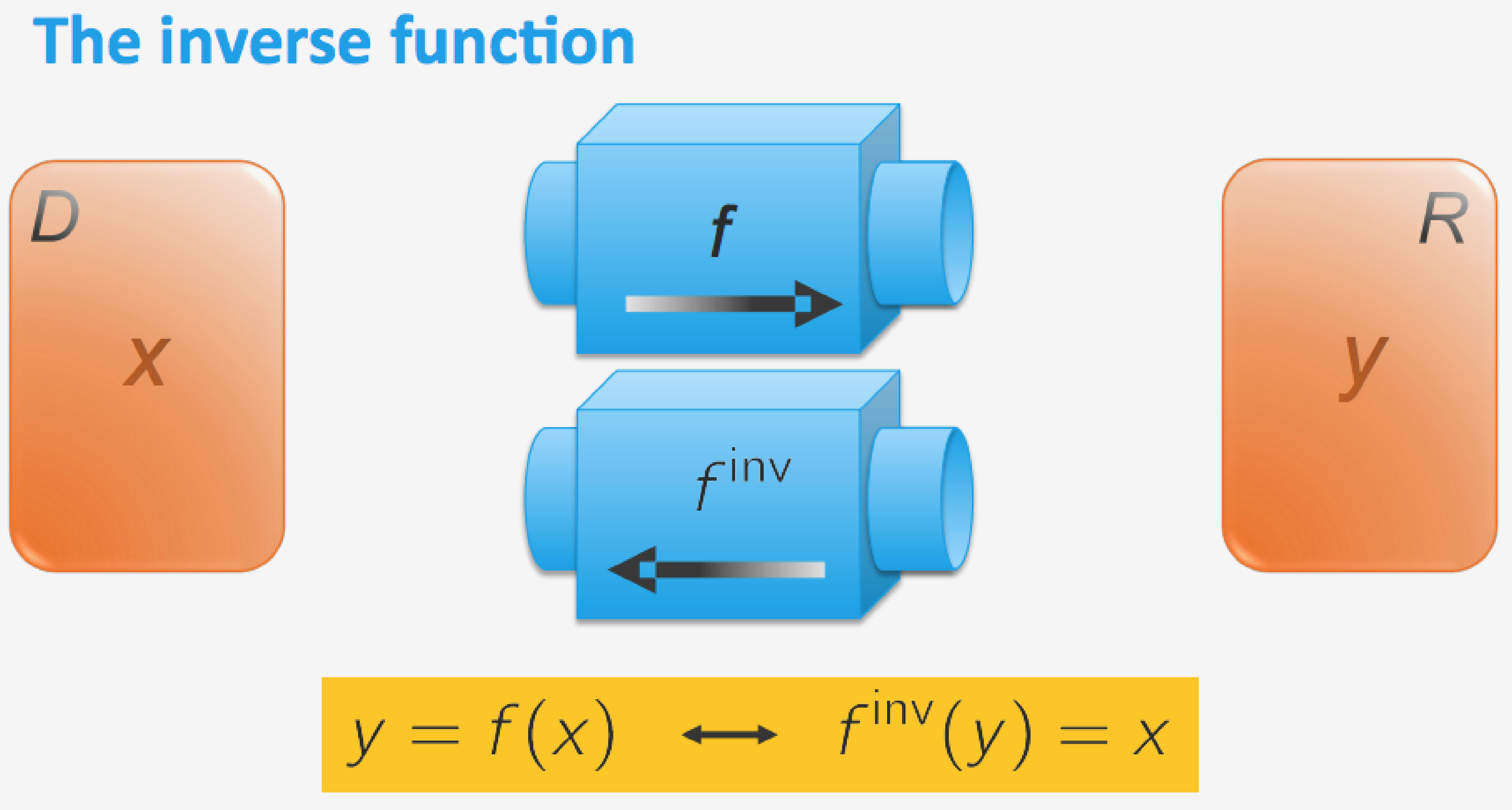
- The translation of Celsius to Fahrenheits and the vice versa **is an example of inverse functions!**



F	C
32	0
41	5
50	10
59	15
68	20
77	25

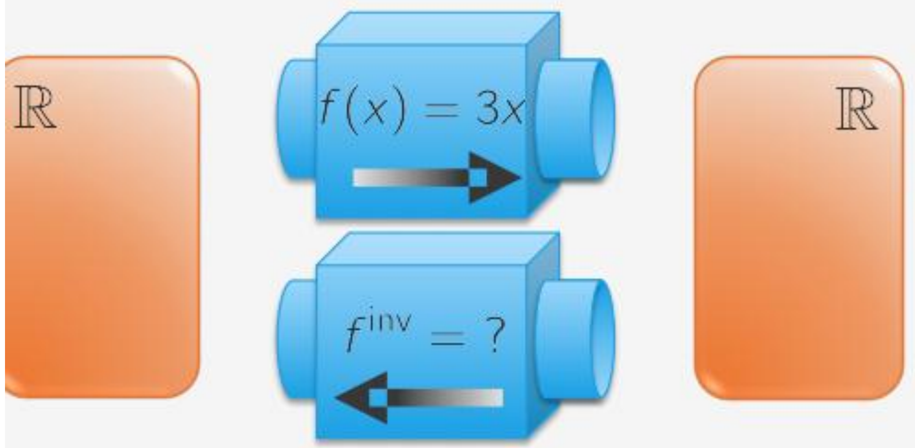
- The domain of *f* is the range of f^{inv}
- The range of *f* is the domain of f^{inv}

The Inverse Functions



The Inverse Functions

Example: $f(x)=3x$



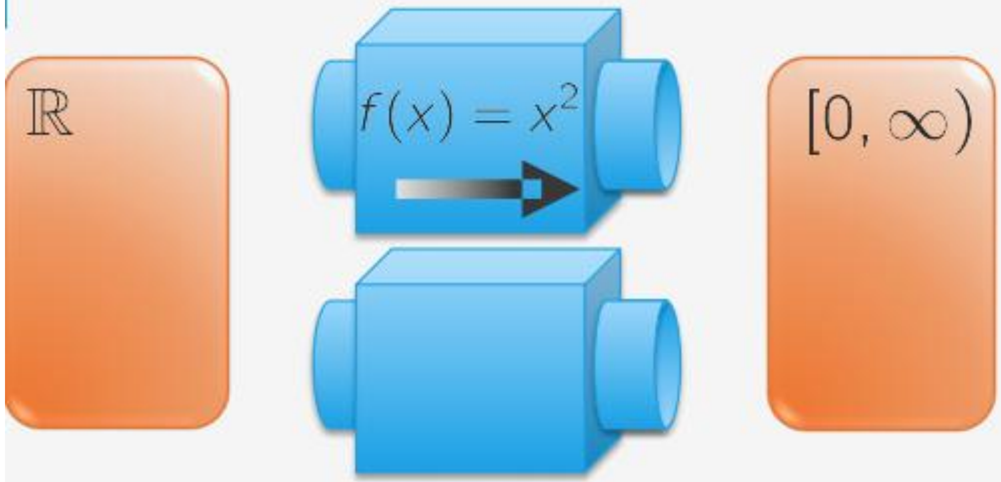
x	y
-2	-6
-1	-3
0	0
1	3
2	6

$y = 3x \longrightarrow x = \frac{1}{3}y$

$\longrightarrow f^{\text{inv}}(x) = \frac{1}{3}x$

The Inverse Functions

Example: $f(x)=x^2$



x	y
-2	4
-1	1
0	0
1	1
2	4

$$y = x^2 \longrightarrow x = \sqrt{y} \text{ or } x = -\sqrt{y}$$

x^2 with domain $\mathbb{R}$ has no inverse function!

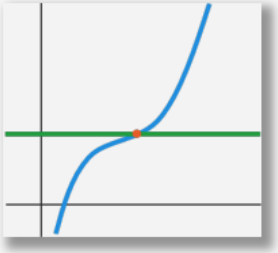
The Inverse Functions

Horizontal line test

f has an inverse ⟷ For each y in the range of f the equation $y = f(x)$ has exactly one solution.

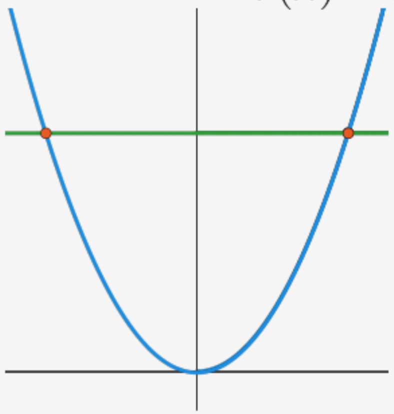
f is injective

Horizontal line test
f is injective if the graph of f intersects any horizontal line in at most one point



Example: $f(x)=x^2$ again

x	y
-2	4
-1	1
0	0
1	1
2	4



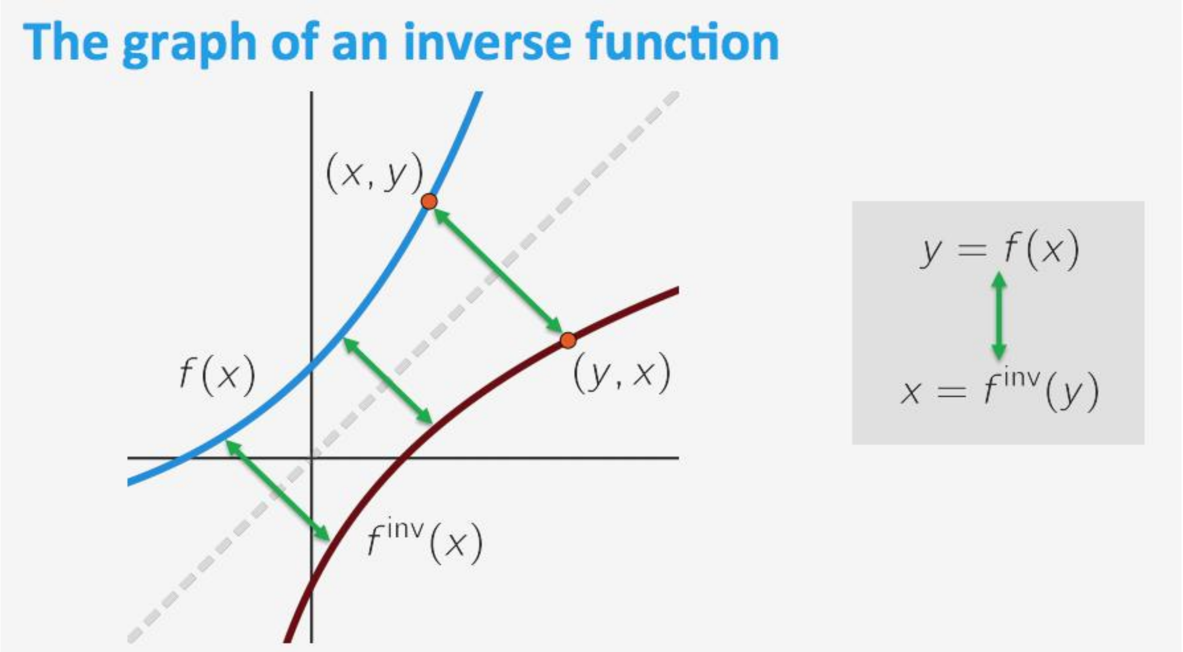
$y = x^2$ ⟶

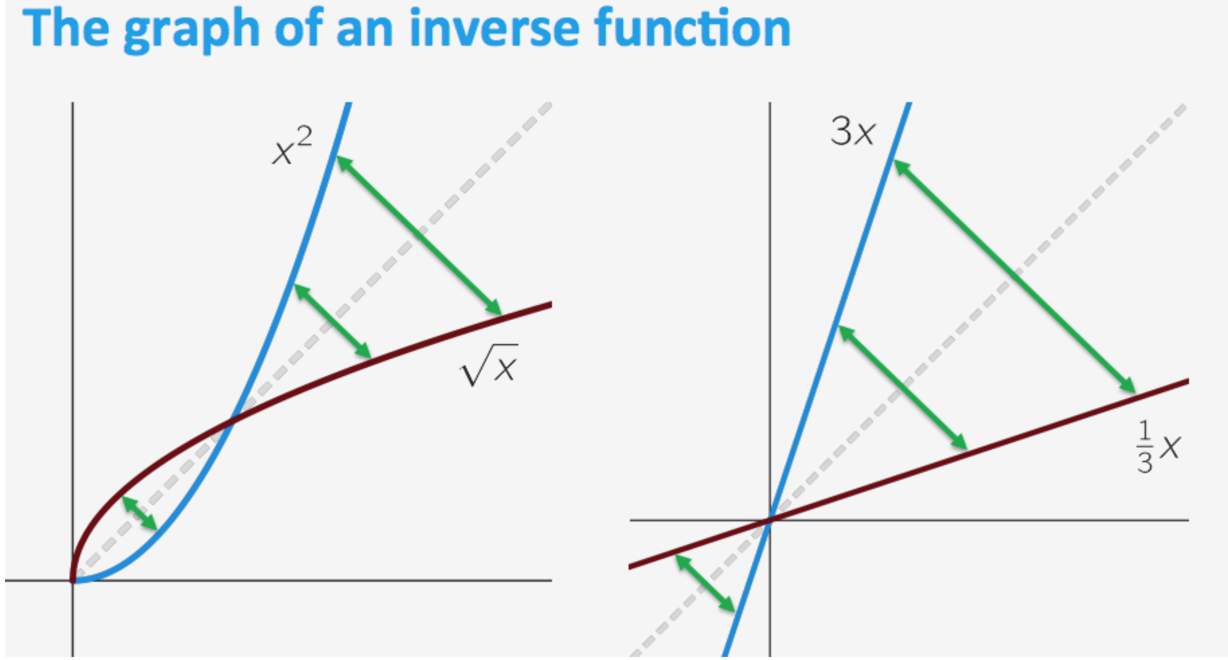
$x = \sqrt{y}$ or $x = -\sqrt{y}$ ✗

$f(x) = x^2$ on $[0, \infty)$

$f^{inv}(x) = \sqrt{x}$

The Inverse Functions





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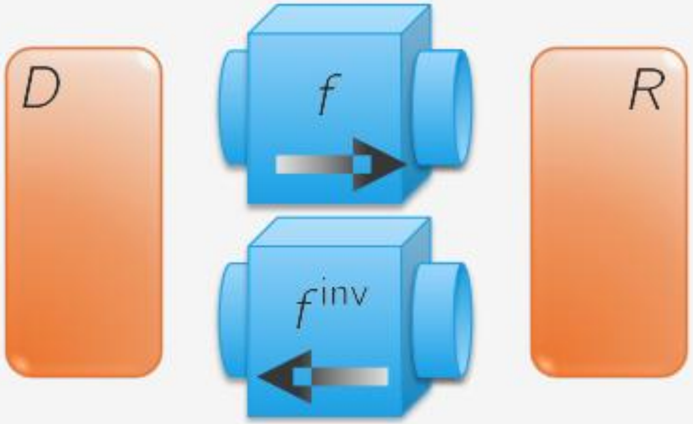
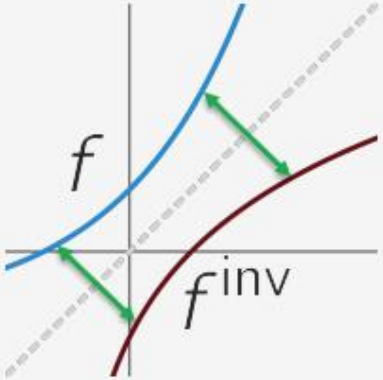
The Inverse Functions

Summary

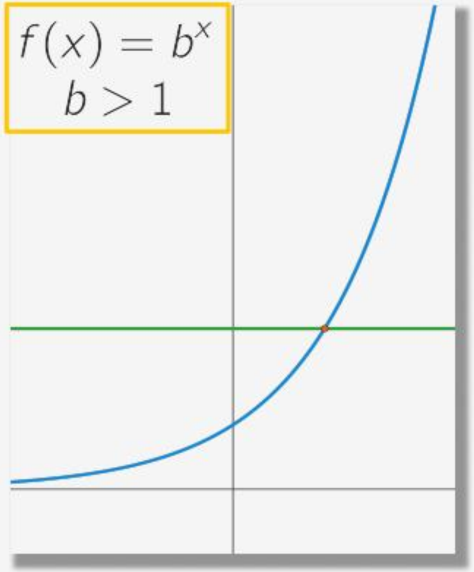
- Inverse exists $\longleftrightarrow$ injective
  - Horizontal line test
- $f^{\text{inv}}(y) = x \longleftrightarrow y = f(x)$
- f^{inv} has domain R and range D
- Graph: reflect in $y = x$



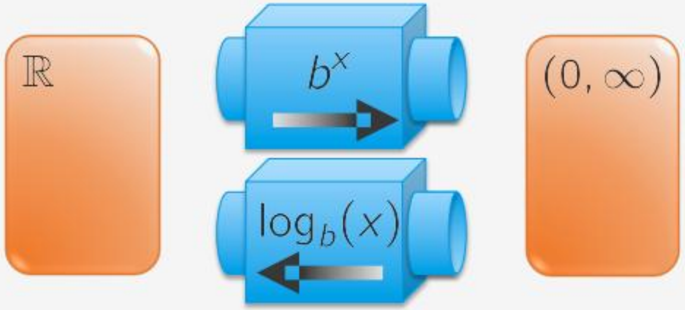



People have 10 fingers and 10 toes. It makes sense to use base 10.
Natural-e.
log
ln
(C)Copyright 2010, C. Burke. All rights reserved. 3/30

The exponential function



$f^{\text{inv}}(x) = \log_b(x)$



$\log_b(x)$ is the inverse of b^x

- $y = \log_b(x) \longleftrightarrow x = b^y$
- **Domain:** $(0, \infty)$
- **Range:** $\mathbb{R}$

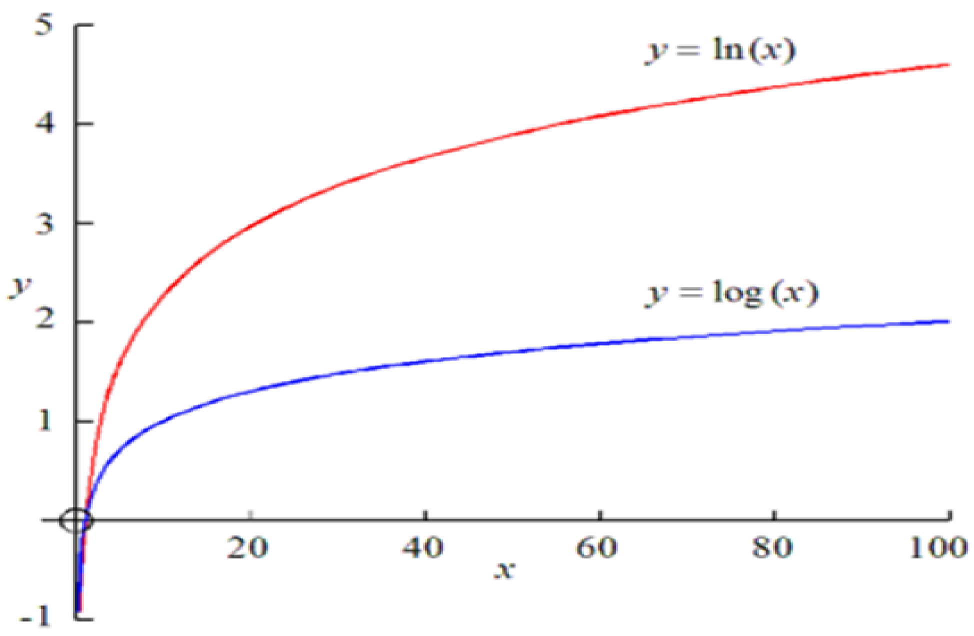
- $\log_2(8) = \log_2(2^3) = 3$
- $\log_5(\frac{1}{5}) = \log_5(5^{-1}) = -1$
- $\log_b(b^a) = a$
- $\log_b(1) = \log(b^0) = 0$

The logarithmic function

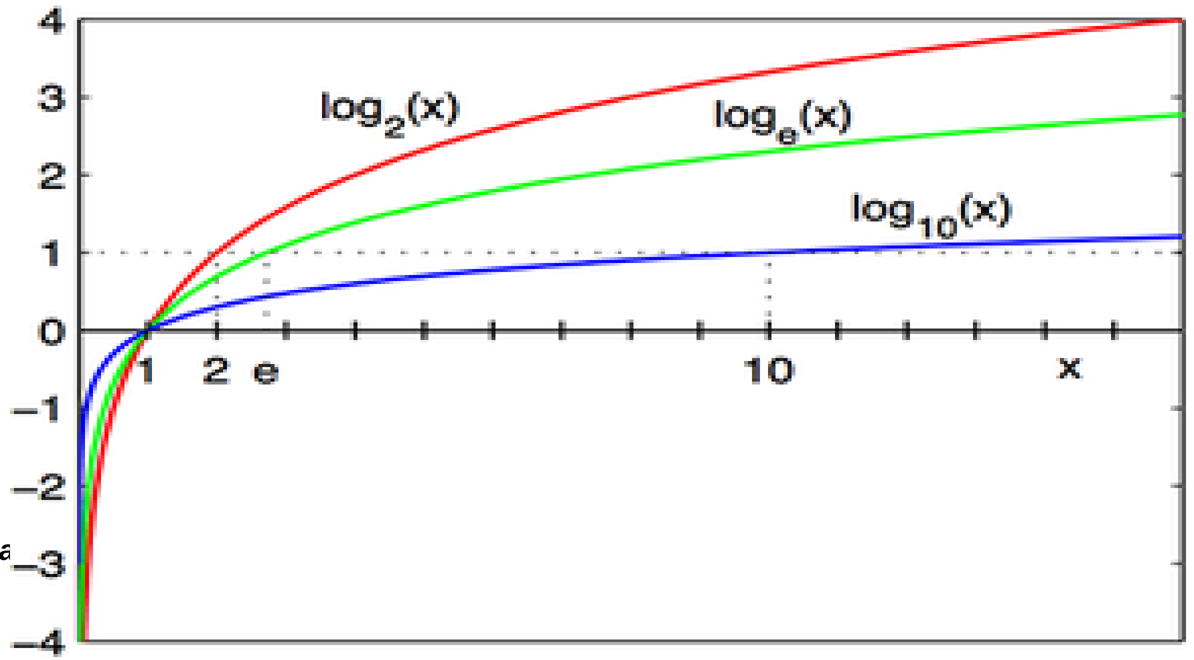
$$y = a^z \Leftrightarrow z = log_a(y)$$

Natural logarithm

$$y = e^z \Leftrightarrow z = ln(y)$$



Other logarithmic functions



The logarithmic function

Properties

1. $log_a(xy) = log_a(x) + log_a(y)$
2. $log_a\left(\frac{x}{y}\right) = log_a(x) - log_a(y)$
3. $log_a(x^b) = b\, log_a(x)$

- $y = \log_b(x) \longleftrightarrow b^y = x$
- $\log_b(x)$ grows slower than x^d for any $d > 0$

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The logarithmic function

Properties – Changing base

$$\log_b(A) = \frac{\log_c(A)}{\log_c(b)} \longleftrightarrow \log_c(A) = \log_c(b)\log_b(A)$$

$$\log_c(A) = \log_c(b)\log_b(A)$$

$$\log_{10}(A) = C\log_2(A)$$

$A = 2:$

$$\log_{10}(2) = C\log_2(2) = C$$

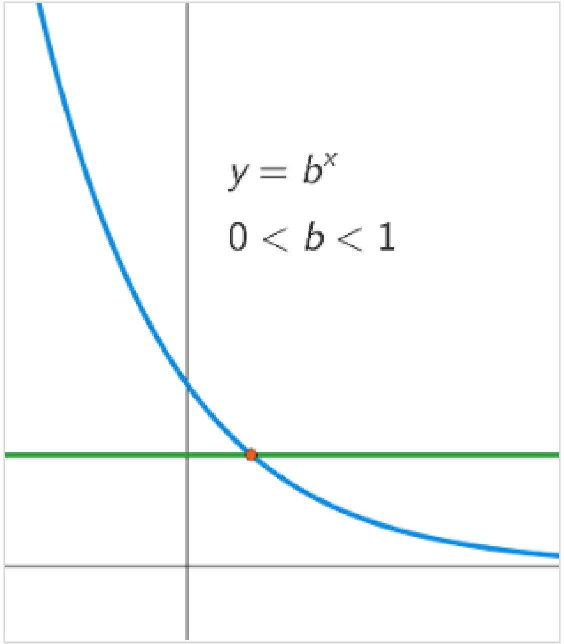
$$\log_{10}(A) = \underbrace{\log_{10}(2)}_{\approx 0.3}\log_2(A)$$

The Logarithmic Function

Explanation: Logarithms in base < 1

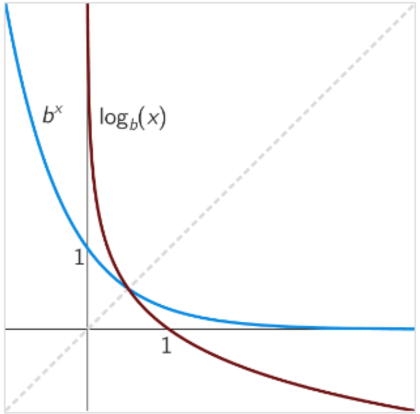
We defined the logarithm $\log_b(x)$ as the inverse function of b^x, where we assumed that $b > 1$. But b^x is also defined for $0 < b \leq 1$.
Assume from here on that $0 < b < 1$. In the figure we see that the graph of b^x satisfies the horizontal line test, so b^x is an injective function and therefore has an inverse function. This inverse function is $\log_b(x)$, but now with $0 < b < 1$. So, just as before,

$$y = \log_b(x) \quad \Leftrightarrow \quad b^y = x.$$



The Logarithmic Function

$\log_{\frac{1}{2}}(2) = -1$, since $\left(\frac{1}{2}\right)^{-1} = 2$



Graph

Since $\log_b(x)$ is the inverse of b^x, the graph of $\log_b(x)$ is obtained from the graph of b^x by reflection in the line $y = x$.

Relation with $\log_{\frac{1}{b}}(x)$

Another way to obtain the graph of $\log_b(x)$ is by relating it to the graph of $\log_{\frac{1}{b}}(x)$. Write x as b^a for some number a, then using rules of calculation for exponential functions gives

$$x = b^a \qquad \Leftrightarrow \qquad x^{-1} = \left(\frac{1}{b}\right)^a.$$

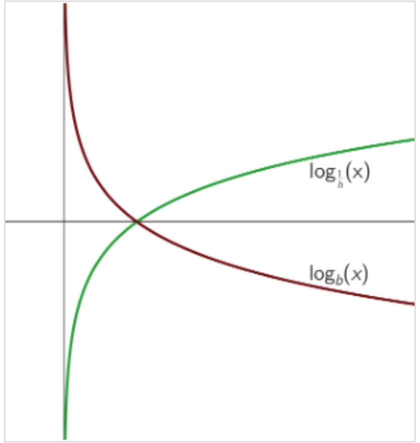
Using the definition of the logarithm, we find

$$\log_b(x) = \log_b(b^a) = a = \log_{\frac{1}{b}}\left(\frac{1}{b}\right)^a = \log_{\frac{1}{b}}(x^{-1}).$$

We take the exponent -1 out of the logarithm to obtain the identity

$$\log_b(x) = -\log_{\frac{1}{b}}(x).$$

(Note that this is a special case of the change of base formula!) So the graph of $\log_b(x)$ is obtained from the graph of $\log_{\frac{1}{b}}(x)$ by reflection in the x-axis.



Complex Numbers

Brief introduction

Imaginary numbers

Some equations (such as $x^2+1 = 0$ for example) cannot be solved if we are assume that the solution x is a real number (we all know that no real number exist such that $x^2=-1$).

This is why mathematicians invented imaginary numbers, to be able to solve equations such as $x^2=-1$. The solution to this equation, is an imaginary number, called the “imaginary unit” *i*. So, we will have $i^2=-1$, or $i=\sqrt{(-1)}$.

Why not make full use of complex numbers to give a “measure of the opposition” that a RLC circuit presents to a current when a voltage is applied… That is, the “total resistance” of RLC circuits!!!

Complex Numbers in Electronic Engineering

Voltage, current, resistance, impedance, A/C circuits analysis, digital signal processing, transients, control systems, digital image processing,

and many many more…

$$x^2+1 = 0$$

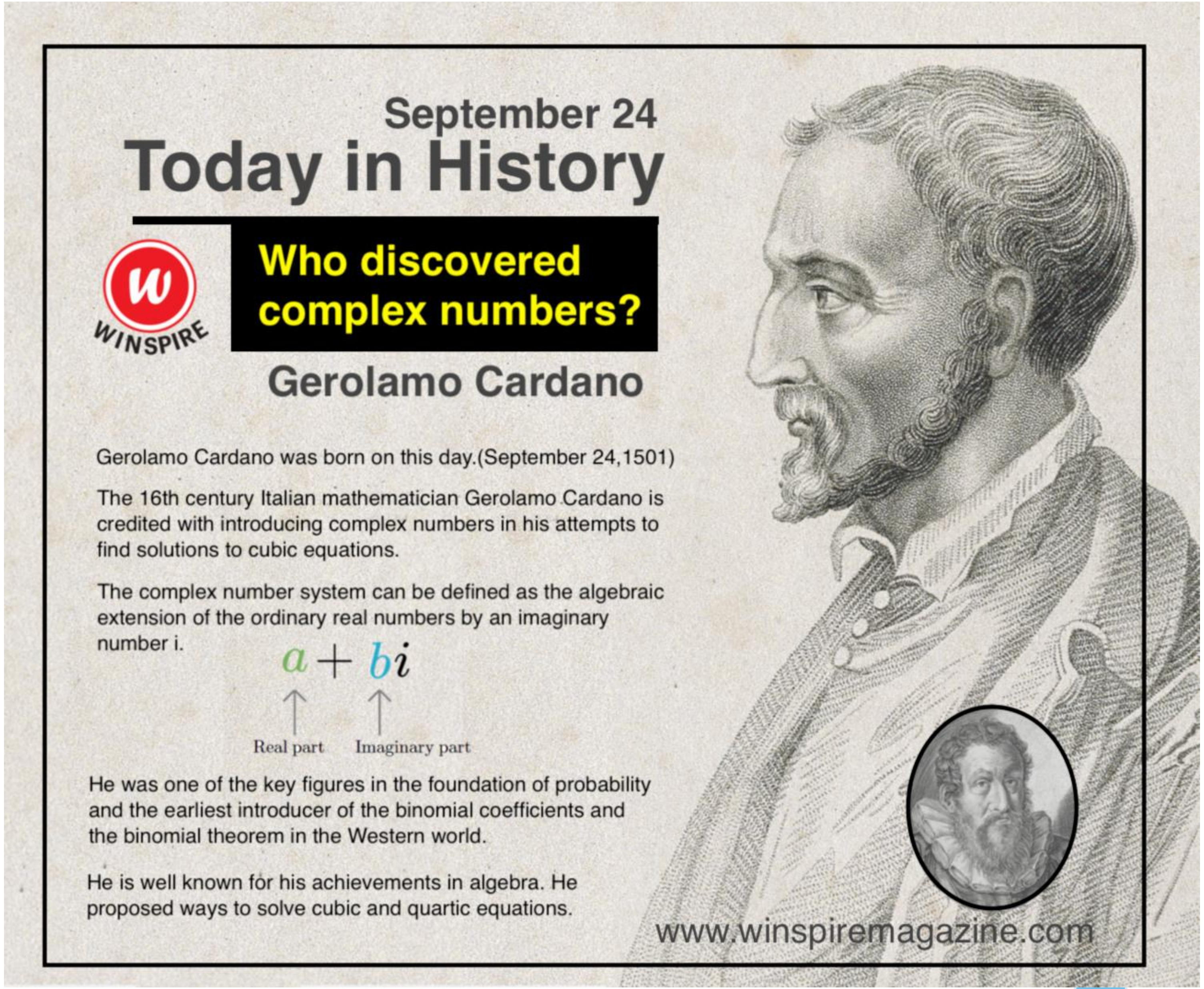
$$x^2=-1$$???

$$x \in \mathbb{R}$$

$$i^2=-1$$

$$i=\sqrt{-1}$$

Complex numbers were first mentioned by the italian mathematician Gerolamo Cardano



Complex numbers

$$\alpha x^2+\beta x+\gamma=0$$

$$\Delta=\beta^2-4\alpha\gamma.$$

$$x_1=\frac{-\beta+\sqrt{\Delta}}{2\alpha}$$

$$x_2=\frac{-\beta-\sqrt{\Delta}}{2\alpha}$$

$\Delta\geq 0$ $\Delta< 0$

$$x^2-2x+4=0$$

$$x_1=1+\sqrt{3}i \quad x_2=1-\sqrt{3}i$$

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Cartesian form

$z \in \mathbb{C}$ $z = a + bi$

$a \in \mathbb{R}$ real part $\mathrm{Re}(z)$

$b \in \mathbb{R}$ imaginary part $\mathrm{Im}(z)$

$$z = 1 + \sqrt{3}i$$

$$Re(z) = 1$$

$$Im(z) = \sqrt{3}$$

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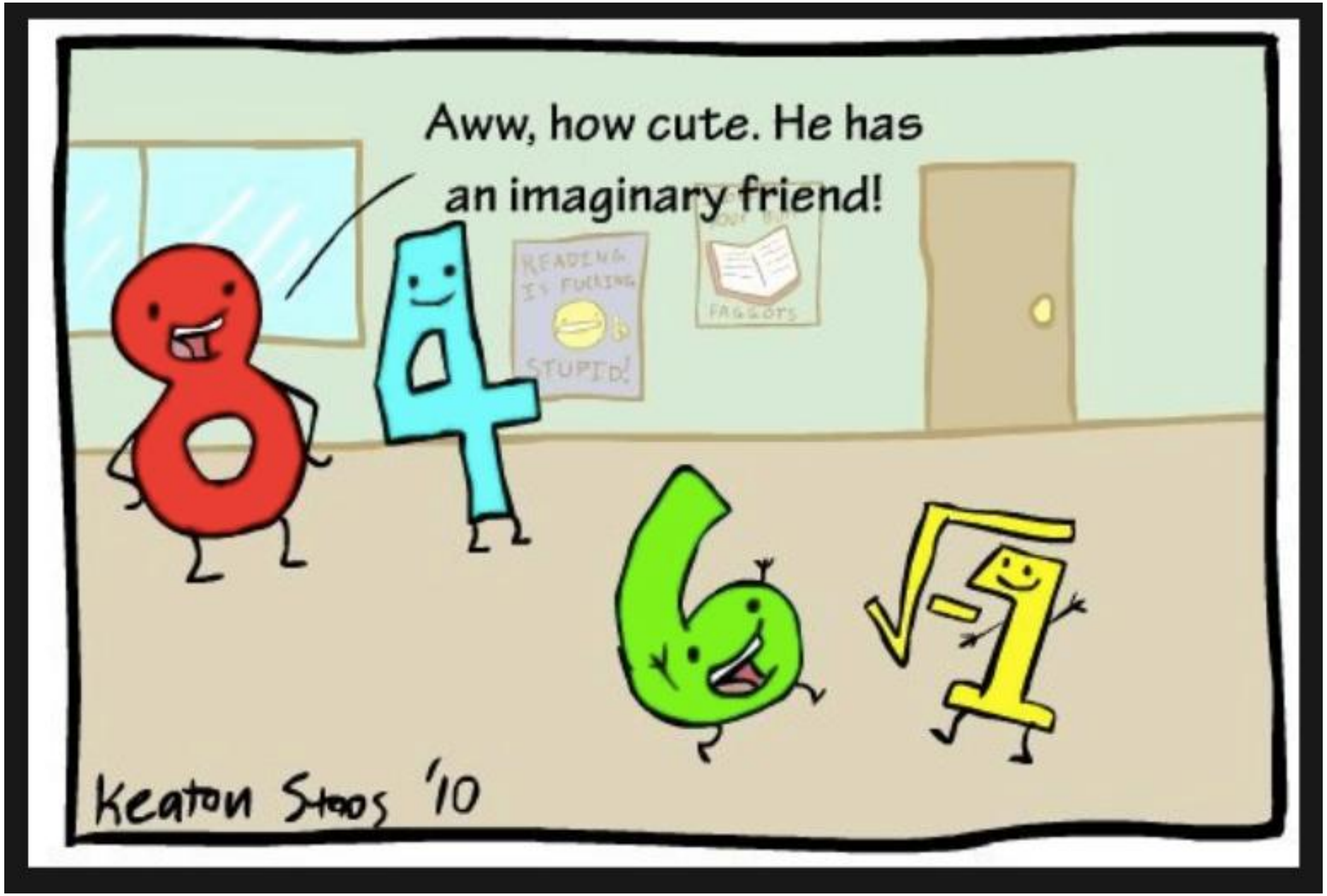
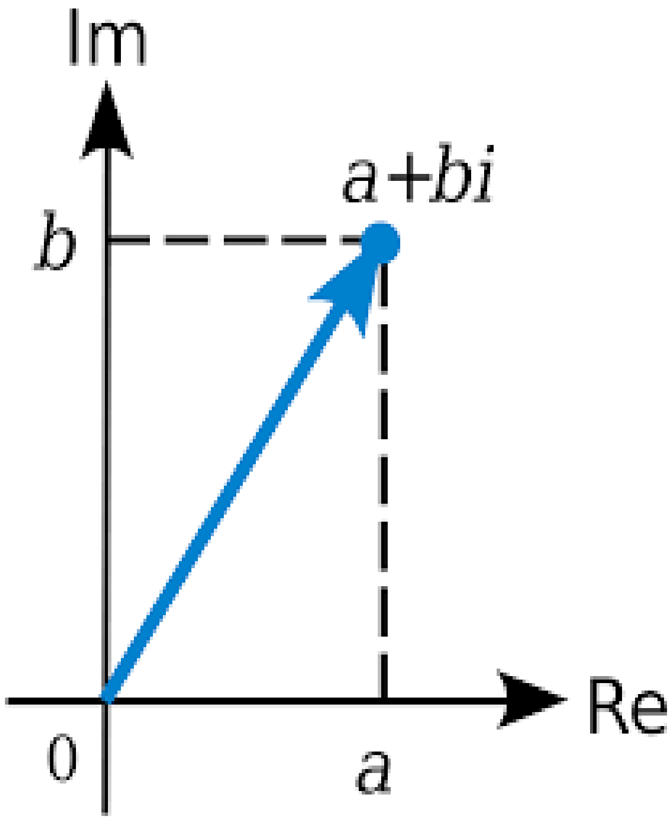


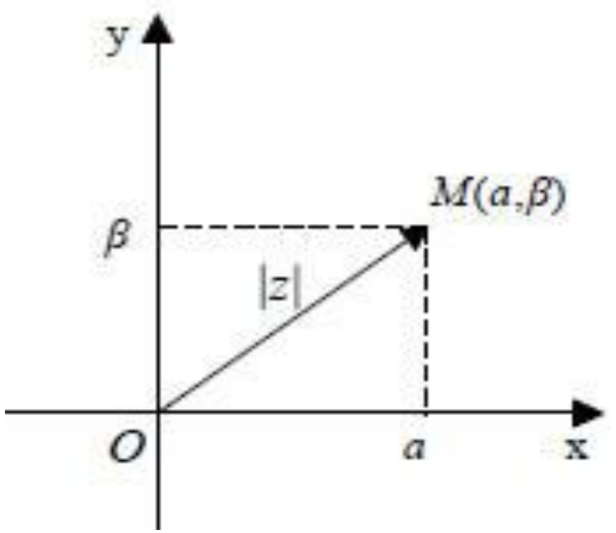
From https://funnyjunk.com/Imaginary+number/funny-pictures/5224095/

Argand graph



$$z = a + bi$$

magnitude or modulus of a complex number



$$|z| = \sqrt{\alpha^2 + \beta^2}$$

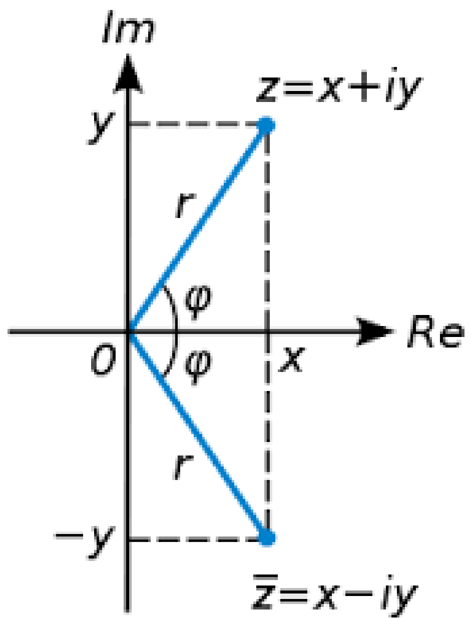
$$z = 1 + \sqrt{3}i$$

$$|z| = 2$$

Conjugate complex number

the **complex conjugate** of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign

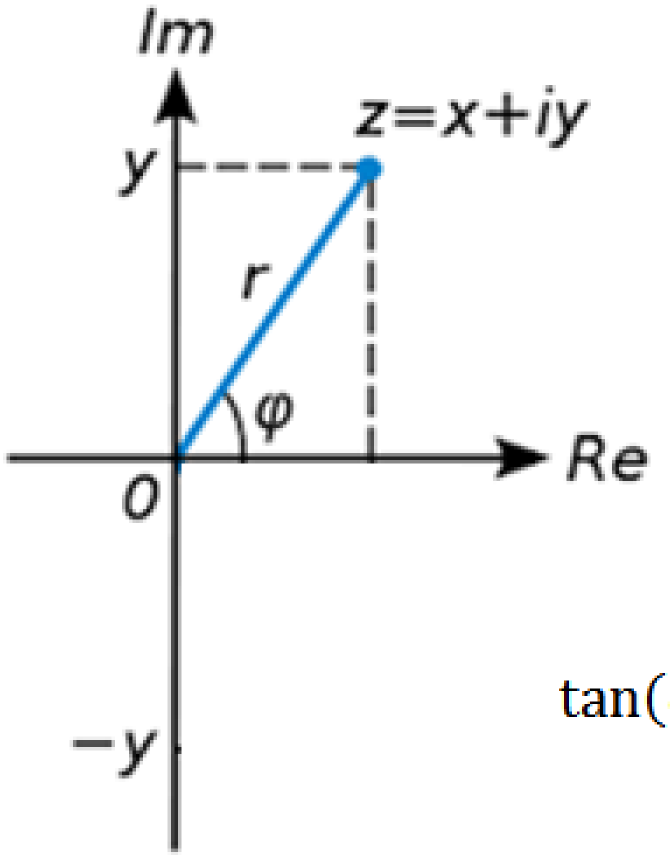
$$z = x + yi$$
$$\bar{z} = x - yi,$$



Example

$$z = 1 + \sqrt{3}i$$
$$\bar{z} = 1 - \sqrt{3}i.$$

Polar form



$$x = |z|\cos(\varphi)$$

$$y = |z|\sin(\varphi)$$

$$z = |z|(\cos(\varphi) + i\sin(\varphi))$$

$$\tan(\varphi) = \frac{y}{x}.$$

Powers of complex numbers, DeMoivre theorem

If $z = x + yi$, or in polar form $z = |z|(\cos(\varphi) + i\sin(\varphi))$, and n positive natural number

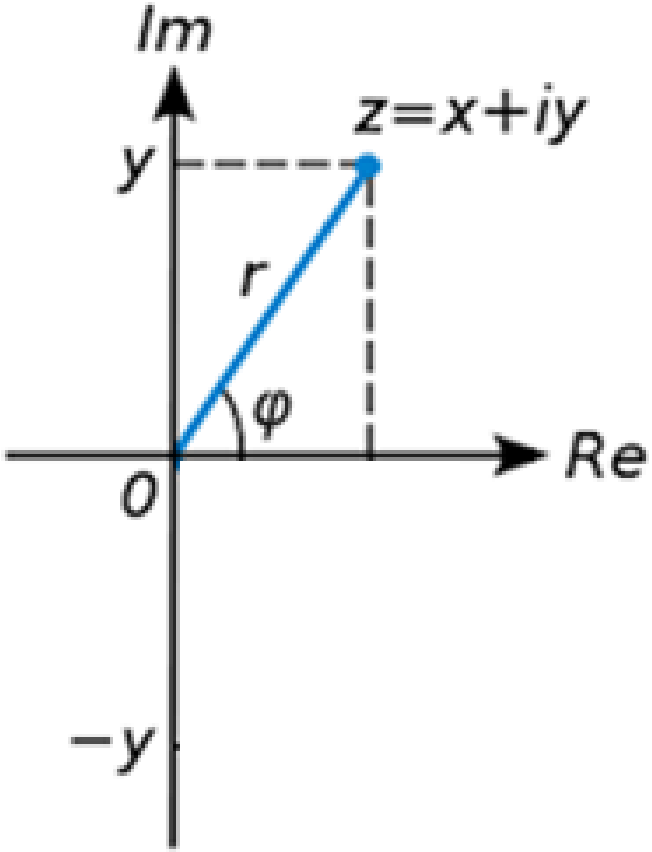
$$z^n = |z|^n(\cos(n\varphi) + i\sin(n\varphi))$$

Example:

$$z = \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2},\ z = \left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right),\ \text{άρα}$$

$$\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)^4 = \cos(\pi) + i\sin(\pi) = -1\,.$$

Euler's formula



$$z = |z|(\cos(\varphi) + i\sin(\varphi))$$

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$$

$$z = |z|e^{i\varphi}$$

Example

$$z = |z|(\cos(\varphi) + i\sin(\varphi))$$

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$$

$$z = \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}$$

$$z = \left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$

$$z = e^{i\frac{\pi}{4}}$$

Euler's formula

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$$

$$e^{-i\varphi} = \cos(-\varphi) + i\sin(-\varphi)$$

$$e^{-i\varphi} = \cos(\varphi) - i\sin(\varphi)$$

$$cos(\varphi) = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$sin(\varphi) = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

Multiplying by j means Rotation by 90º!

$$i^0 = 1$$

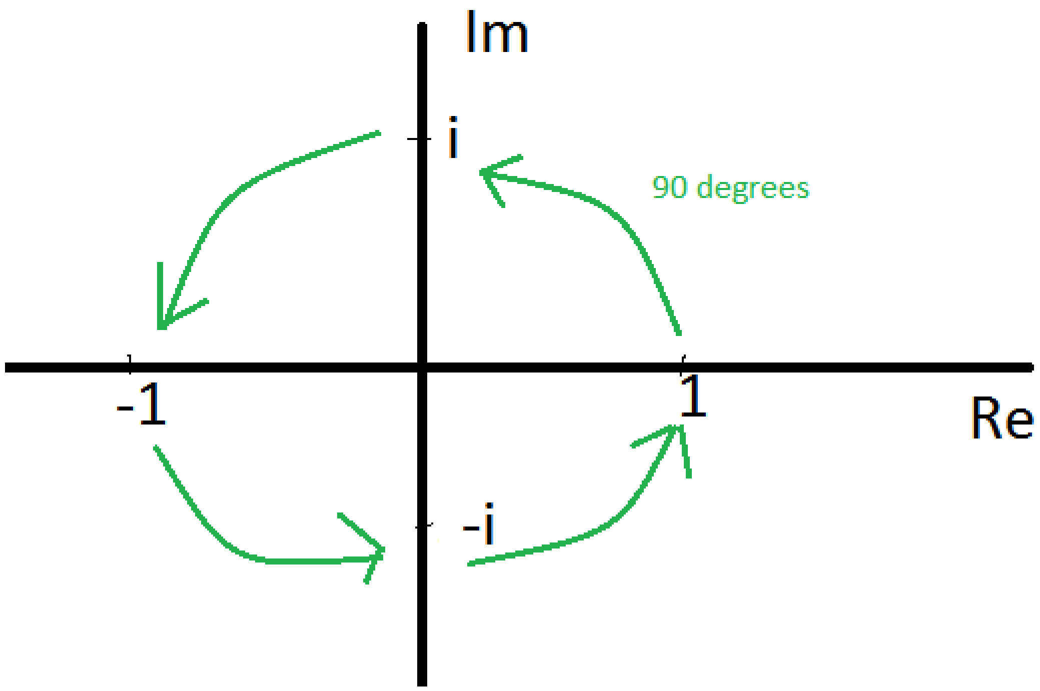
$$i^1 = i$$

$$i^2 = -1$$

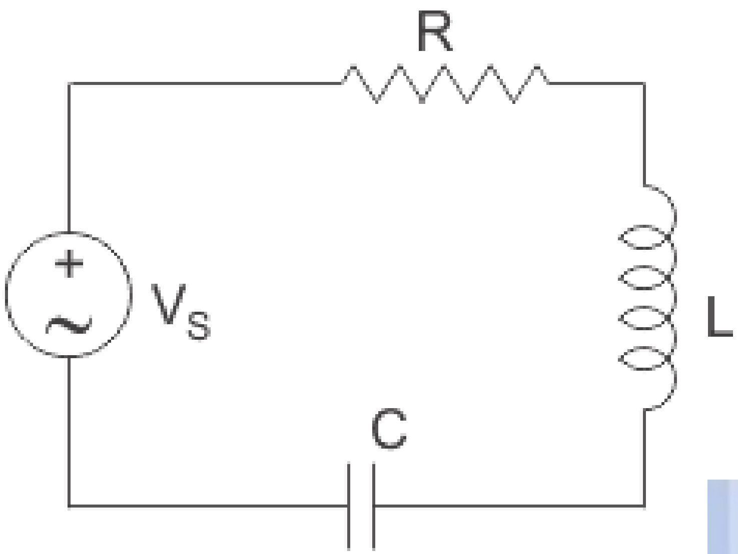
$$i^3 = i \cdot i^2 = (-1)i = -i$$

$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$$

$$i^5 = i^4 \cdot i = 1 \cdot i = i$$



$$i^{-1} = \frac{1}{i} = \frac{1}{i}\frac{i}{i} = \frac{i}{i^2} = \frac{i}{(-1)} = -i$$



In applications in the area of electronic engineering the imaginary unit becomes
j

$$j = \sqrt{-1}$$

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi) \qquad z = |z|e^{j\varphi}$$

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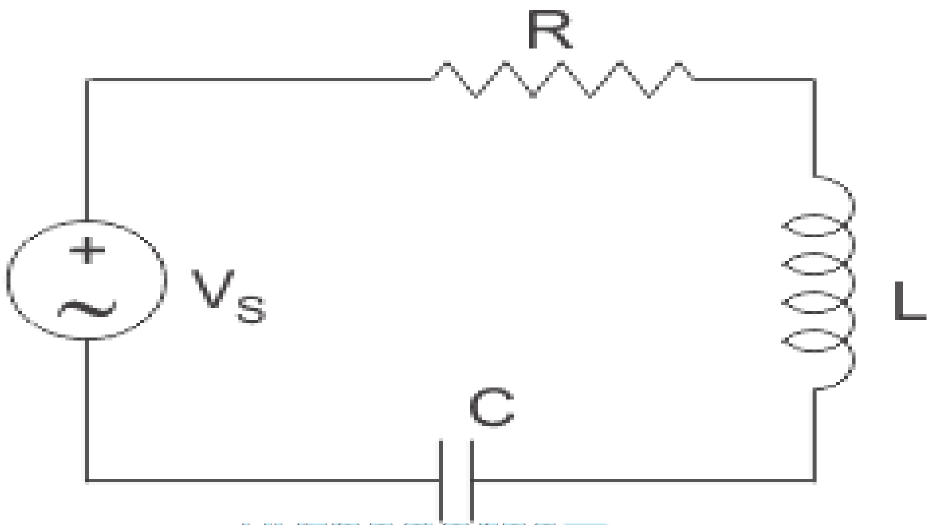
Can we apply complex numbers to find the "total resistance" in an A/C electric circuit?

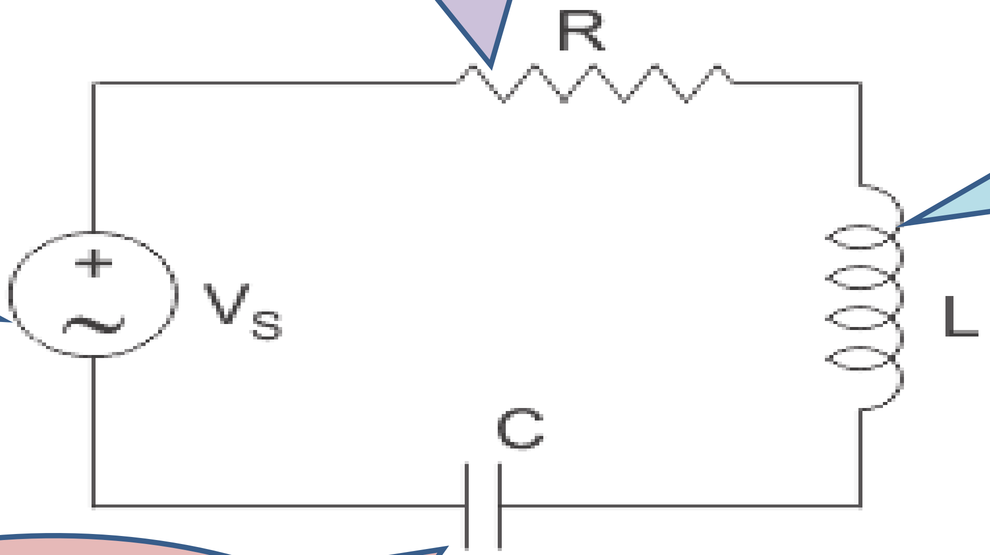
Maybe we can do much more by use of complex numbers?

Keywords:
Complex numbers
A/C electric circuit
Resistance, Impedance



How does a resistance (R) react to A/C current?

A/C input voltage – supplies A/C current to the circuit



How does a inductor (L) react to A/C current?

How does a capacitor (C) react to A/C current?

Experience: The A/C voltage through each element either lags or leads the current… SO???

Impedance - Resistor

$$V = IR$$

I, V sinusoid

$$I = e^{j\omega t}$$

$$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

$$V = Re^{j\omega t}$$

$$\frac{V}{I} = R$$

$$\sin(\omega t) = \frac{1}{2i}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

Impedance - Inductor

I, V sinusoid

$$V = L\frac{dI}{dt}$$

$$I = e^{j\omega t}$$

$$V = L\frac{d(e^{j\omega t})}{dt} = j\omega L\, e^{j\omega t}$$

$$\frac{V}{I} = \frac{j\omega L\, e^{j\omega t}}{e^{j\omega t}} = j\omega L$$

Impedance - Capacitor

$$I = C\frac{dV}{dt}$$

I, V sinusoid

$$V = e^{j\omega t}$$

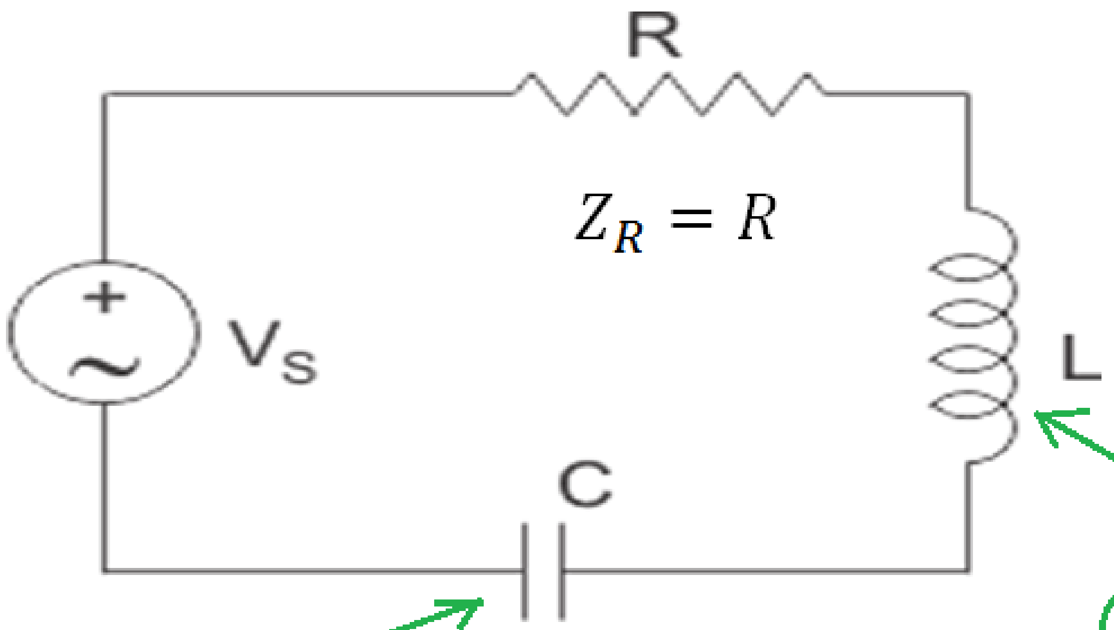
$$I = C\frac{d(e^{j\omega t})}{dt} = j\omega C\, e^{j\omega t}$$

$$\frac{V}{I} = \frac{e^{j\omega t}}{j\omega C\, e^{j\omega t}} = \frac{1}{j\omega C}$$

Total impedance RLC

$$\frac{V}{I} = Z$$

$$I = I_0 \cos(\omega t + \varphi)$$



$$Z_R = R$$

$$Z_L = j\omega L$$

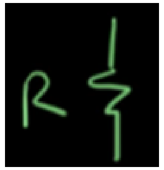
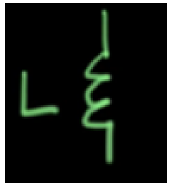
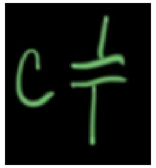
V leads I by 90^o

$$Z_C = \frac{1}{j\omega C}$$

I leads V by 90^o

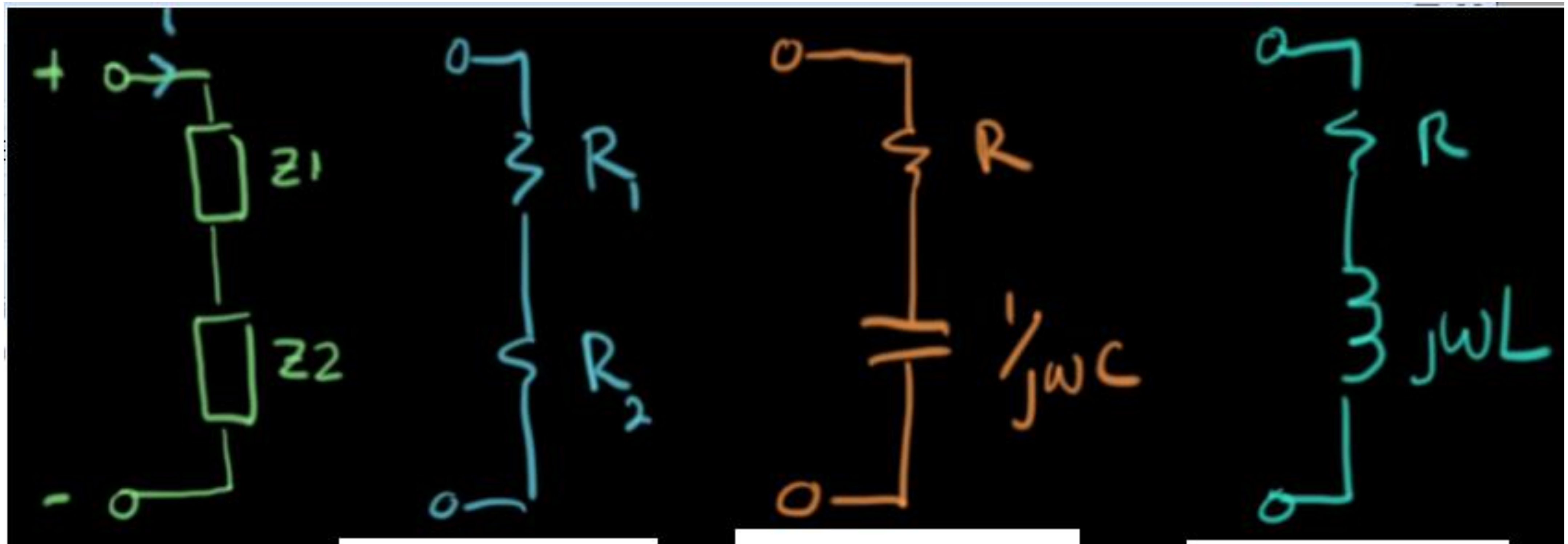
Z

$$\frac{V}{I} = Z$$

ω	0	low	high	∞
 $Z_R = R$	R	R	R	R
 $Z_L = j\omega L$ $\lvert Z_L \rvert = \omega L$	0	low	high	∞
 $Z_C = \frac{1}{j\omega C}$ $\lvert Z_C \rvert = \frac{1}{\omega C}$	∞	high	low	0

Impedance in circuits

$$\frac{V}{I} = Z$$



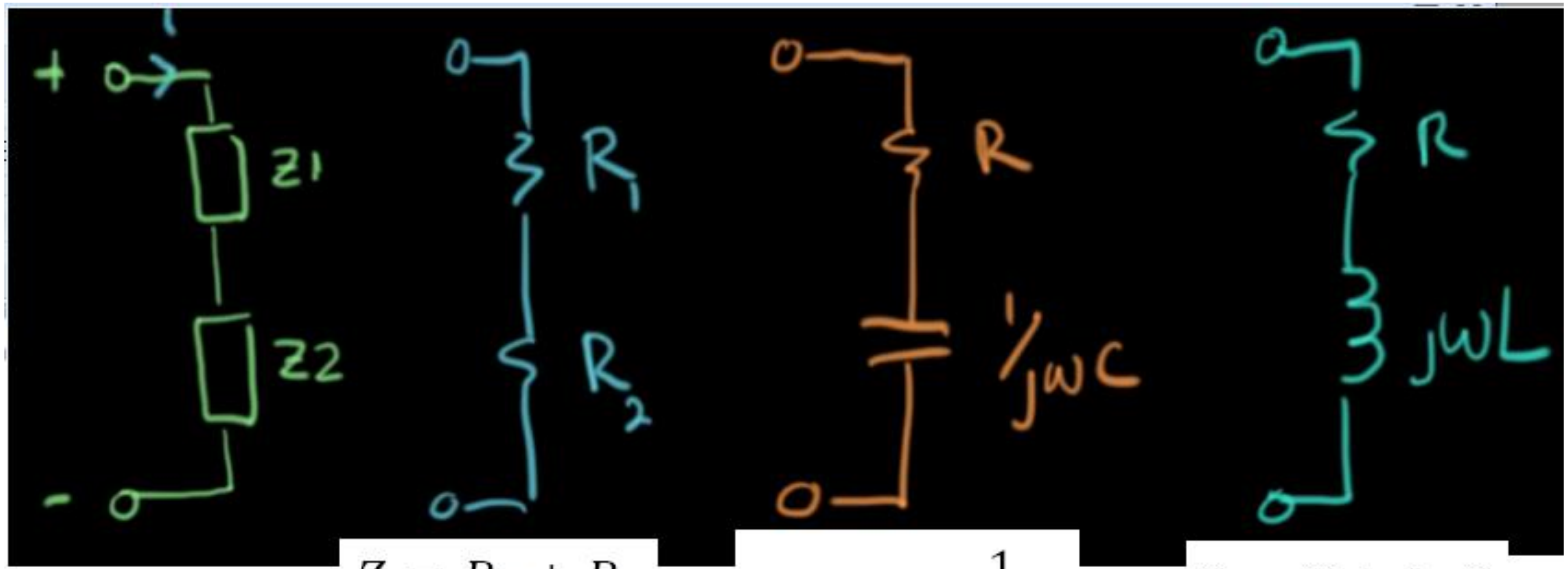
$$Z = R_1 + R_2$$

$$Z = R + \frac{1}{j\omega C}$$

$$Z = R + j\omega L$$

Impedance in circuits

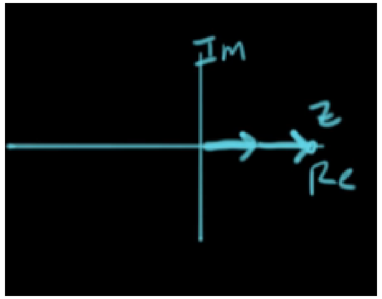
$$\frac{V}{I} = Z$$

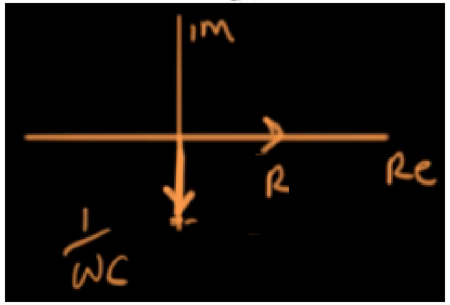


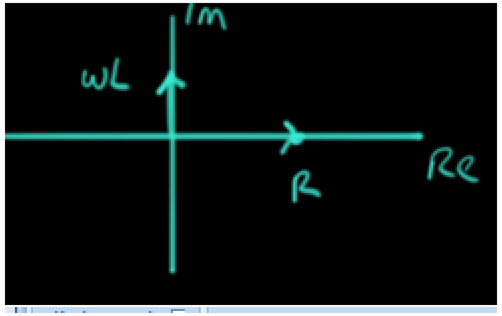
$$Z = R_1 + R_2$$

$$Z = R + \frac{1}{j\omega C}$$

$$Z = R + j\omega L$$







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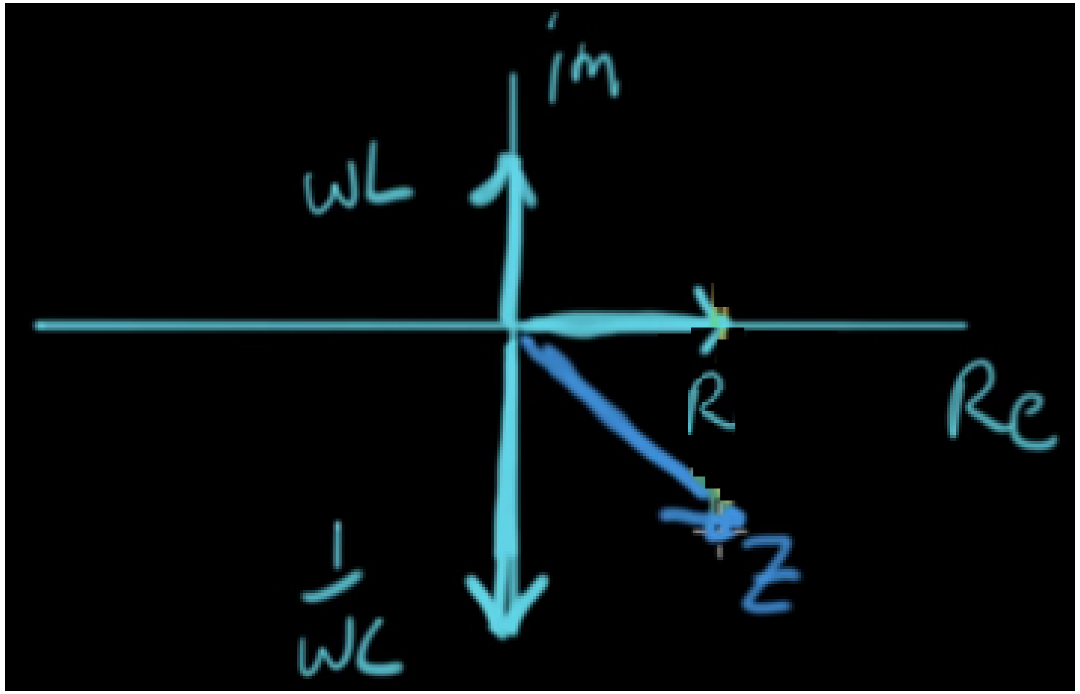
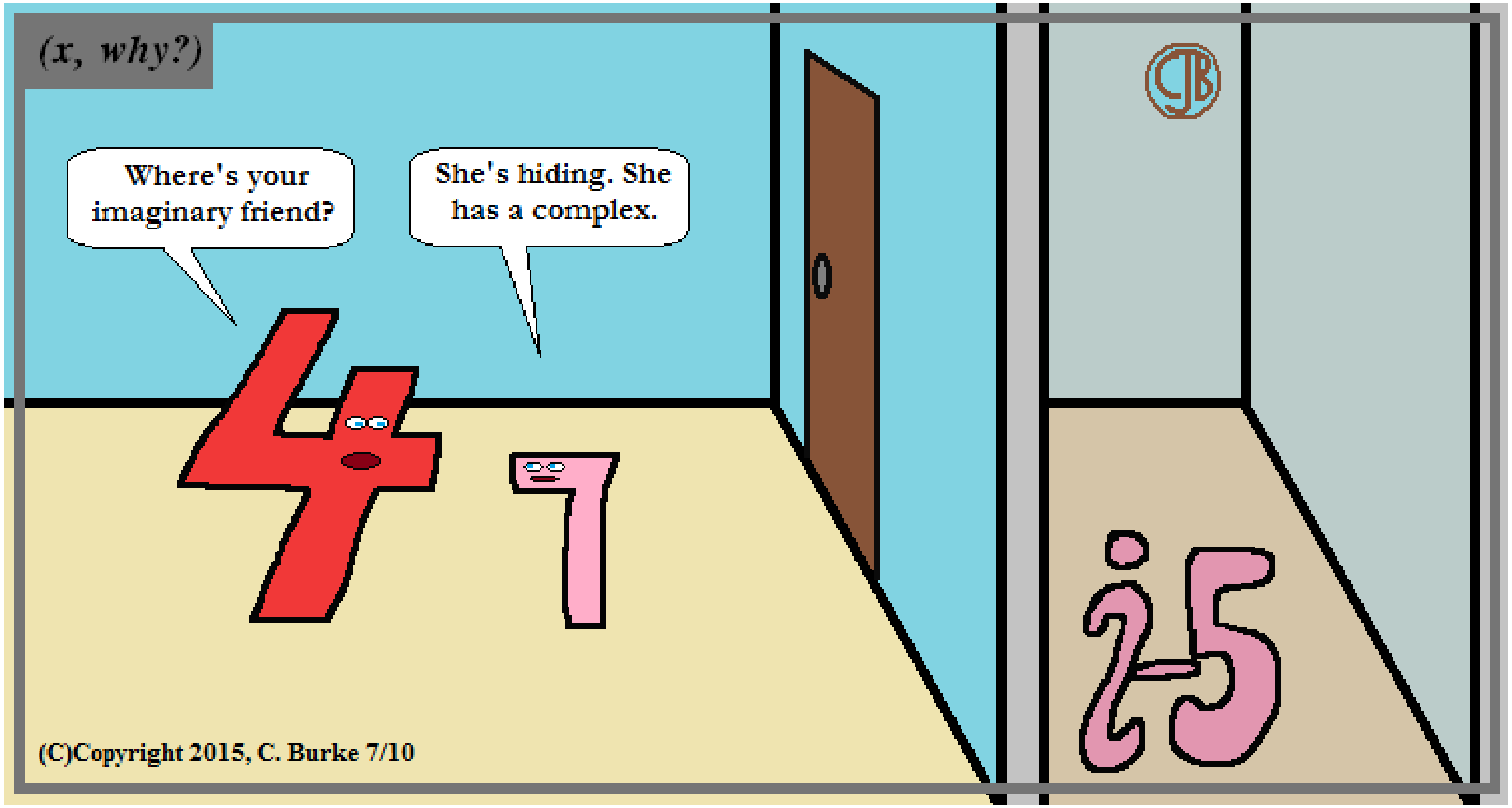
Impedance in circuits

$$Z = R + j\omega L + \frac{1}{j\omega C}$$



$$\text{Impedance, } Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$



From http://mrburkemath.blogspot.com/2015/07/

Chapter Five: Differentiation

An Introduction

The aim of this chapter is: In this chapter we will learn **the definition of a derivative**. This can be used to calculate speed and to obtain the slope of a tangent line for instance. Then we will **discuss standard derivatives and rules of calculation**, followed by exercises to give you the chance to apply these rules. We will also discuss the concept of differentiability and the use of differentiation in optimization problems.

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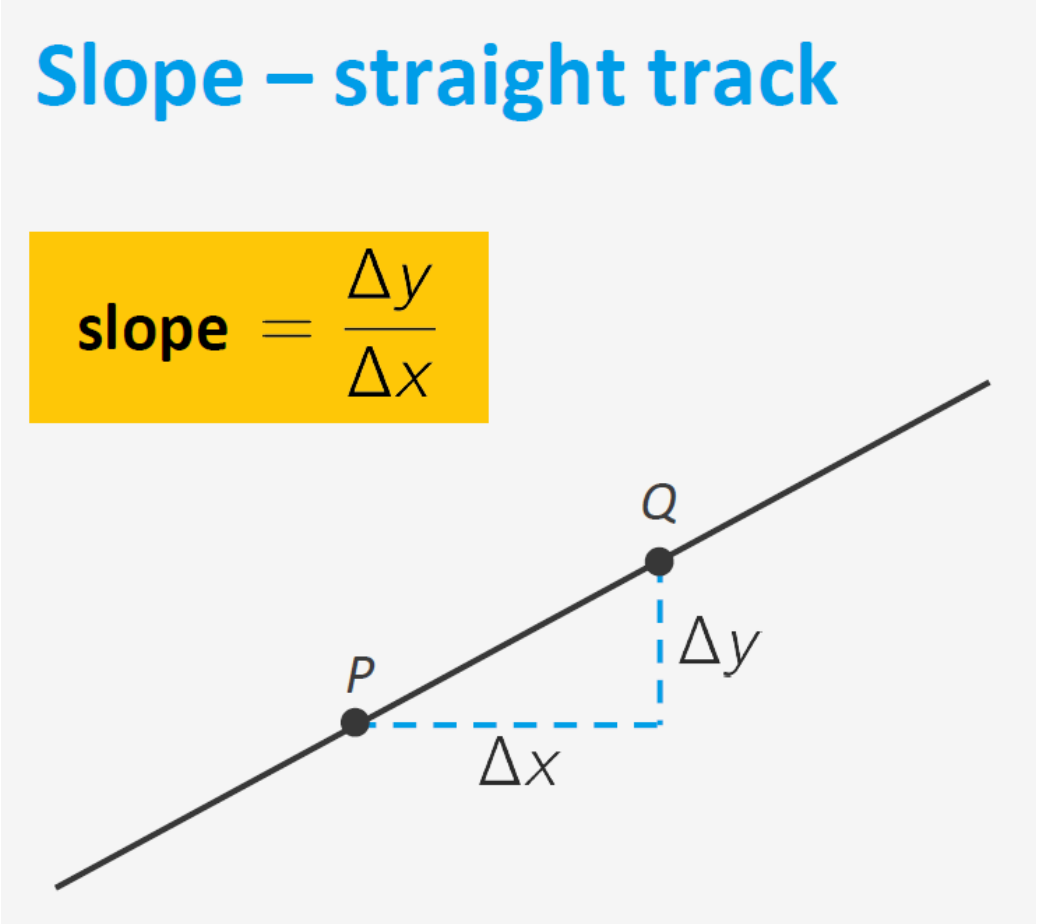
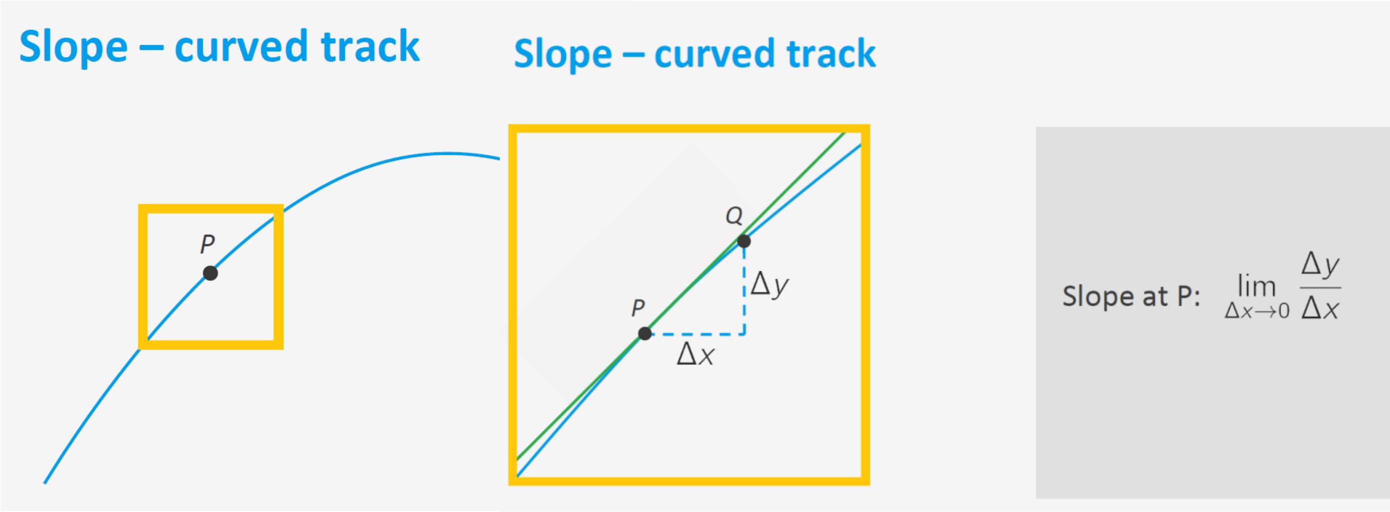
Definition of the Differentiation

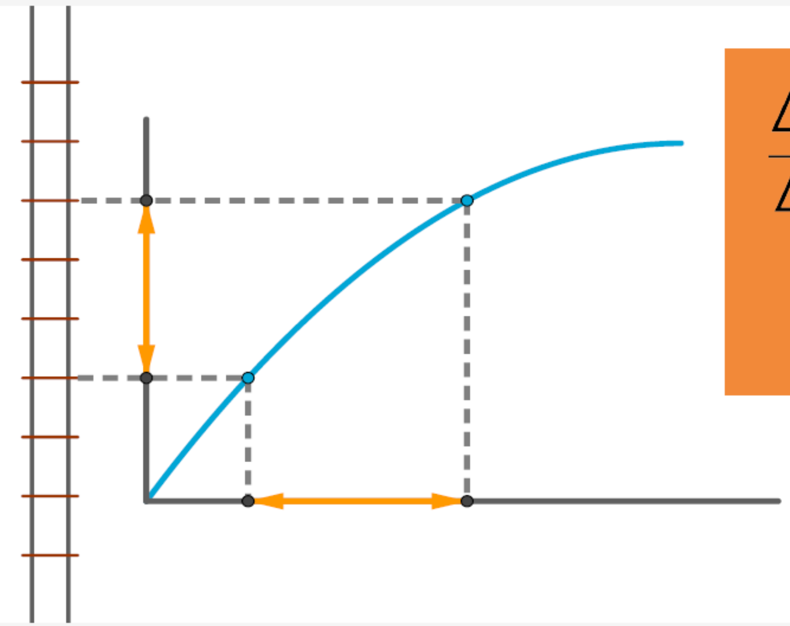
- The differentiation as a mathematical action is linked **with the optimization process**
- The differentiation is related **with the slope** of a function; in case of a straight track the slope of the function (the rate of its change with distance or with time) is constant; whereas in the case of a non straight track the slope changes from point to point. In the latter case we calculate the slope of a point!!!
- The differentiation is linked with the calculation of very important quantities e.g. the speed and the acceleration of a body

Definition of the Differentiation

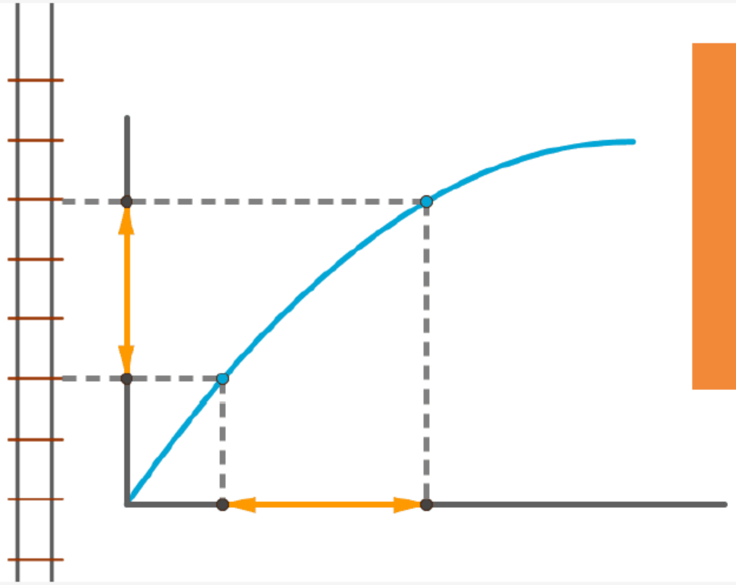




Definition of the Differentiation



$$\frac{\Delta x}{\Delta t} = \textbf{Average speed} \approx \text{Speed at } t$$



$$\text{Speed at } t = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

Definition of the Differentiation

Limits

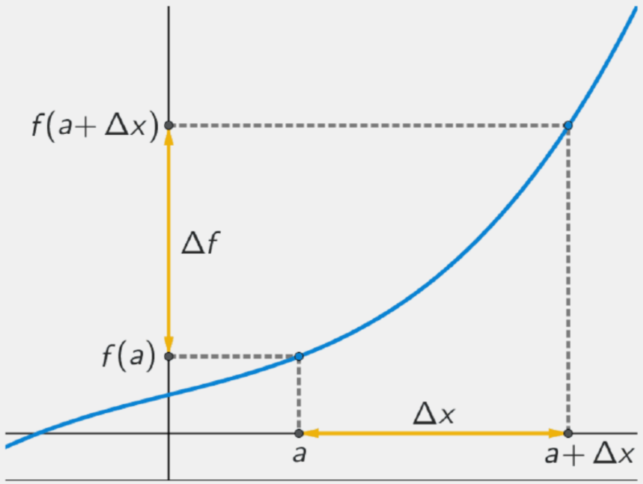
- Slope = $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
- Speed = $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$

Slope and speed are derivatives!

- Slope at position $x = a$: $\frac{dy}{dx}(a)$
- Speed at time $t = a$: $\frac{dx}{dt}(a)$

Definition of the Differentiation

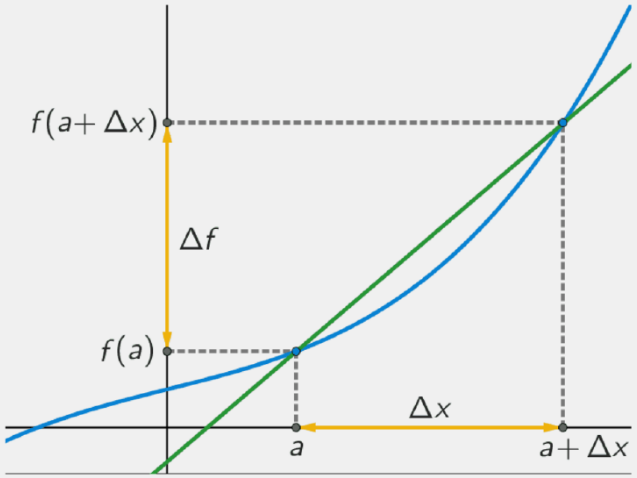
Differentiation – the definition



Difference quotient:

$$\frac{f(a+\Delta x)-f(a)}{\Delta x}$$

Differentiation – geometrical meaning

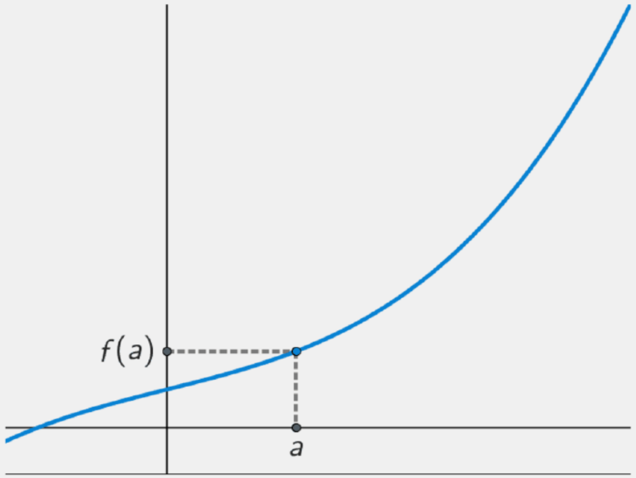


Slope of line *PQ*:

$$\frac{f(a+\Delta x)-f(a)}{\Delta x}$$

Definition of the Differentiation

Differentiation – the definition



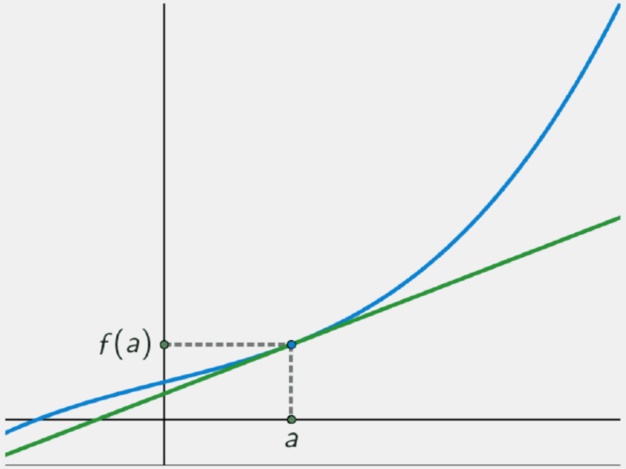
The derivative of f at $x = a$:

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

Notation:

$f'(a)$ or $\frac{df}{dx}(a)$

Differentiation – geometrical meaning



Slope of line PQ:

$$\frac{f(a+\Delta x) - f(a)}{\Delta x}$$

In limit:

Tangent line at *P*

Slope: $f'(a)$

Definition of the Differentiation

Limits

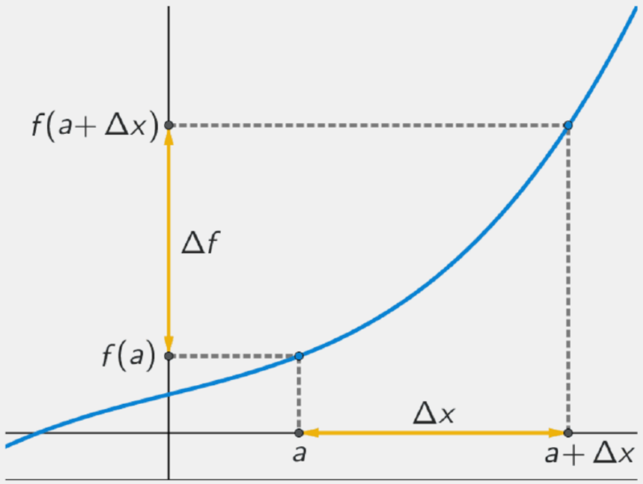
- Slope = $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
- Speed = $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$

Slope and speed are derivatives!

- Slope at position $x = a$: $\frac{dy}{dx}(a)$
- Speed at time $t = a$: $\frac{dx}{dt}(a)$

Definition of the Differentiation

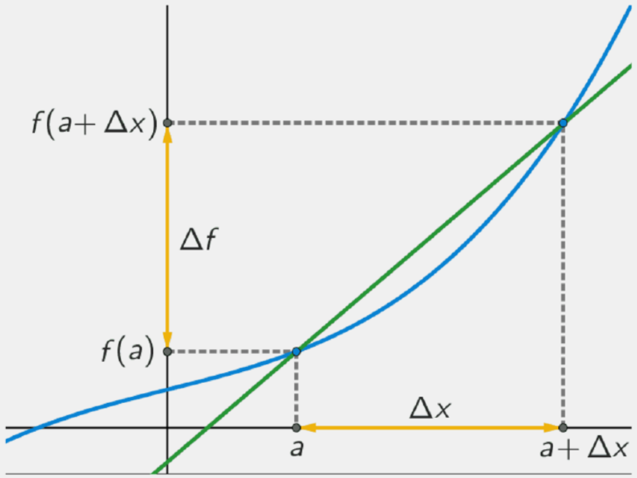
Differentiation – the definition



Difference quotient:

$$\frac{f(a+\Delta x)-f(a)}{\Delta x}$$

Differentiation – geometrical meaning



Slope of line *PQ*:

$$\frac{f(a+\Delta x)-f(a)}{\Delta x}$$

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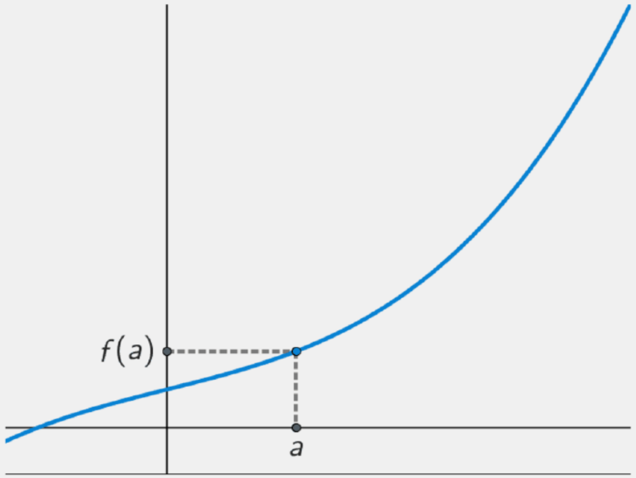
Definition of the Differentiation

Differentiation – the definition



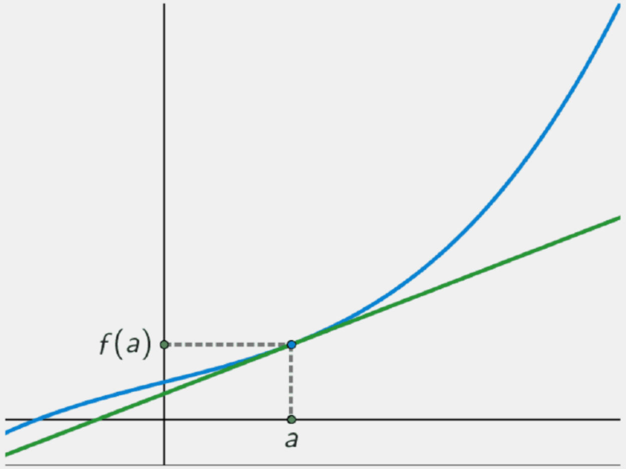
The derivative of f at $x = a$:

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

Notation:

$f'(a)$ or $\dfrac{df}{dx}(a)$

Differentiation – geometrical meaning



Slope of line PQ:

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

In limit:
Tangent line at *P*
Slope: $f'(a)$

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Standard Derivatives & Rules of Calculation

- First, you learn **the derivatives of the standard functions**. Second, you learn **rules to calculate the derivative of combinations of standard functions**. Important rules of calculation are **the product rule and the chain rule**

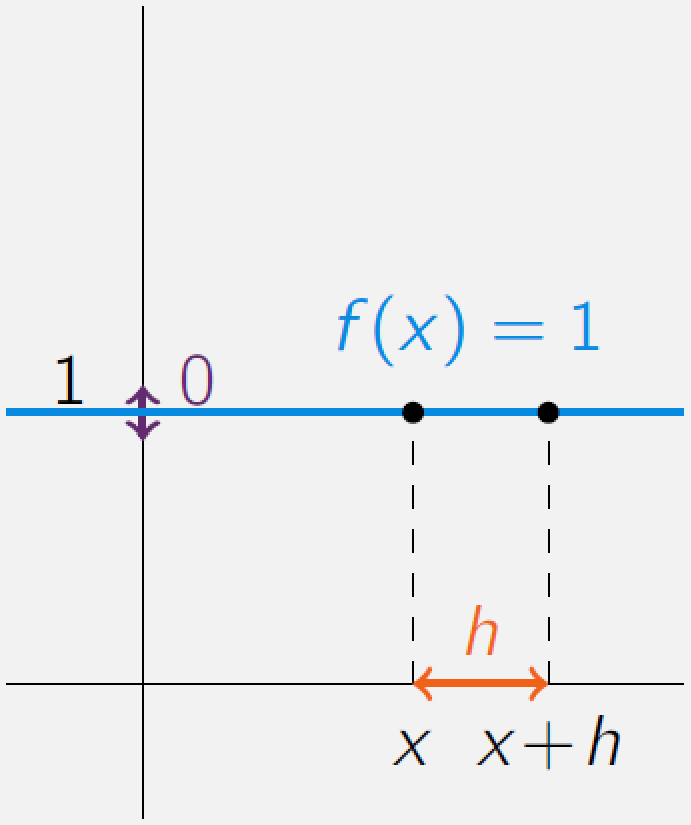
- The way we learn how to calculate the derivative of any function, independently how complicate could be, is the following:

1. Learn the derivative of the standard functions
2. Apply them and in combination of few rules of calculation, derive the derivatives of combinations of these functions

Standard Derivatives & Rules of Calculation

$$f(x) = x^0 = 1$$

$$\frac{f(x+h) - f(x)}{h} = \frac{1-1}{h}$$
$$= \frac{0}{h} = 0$$



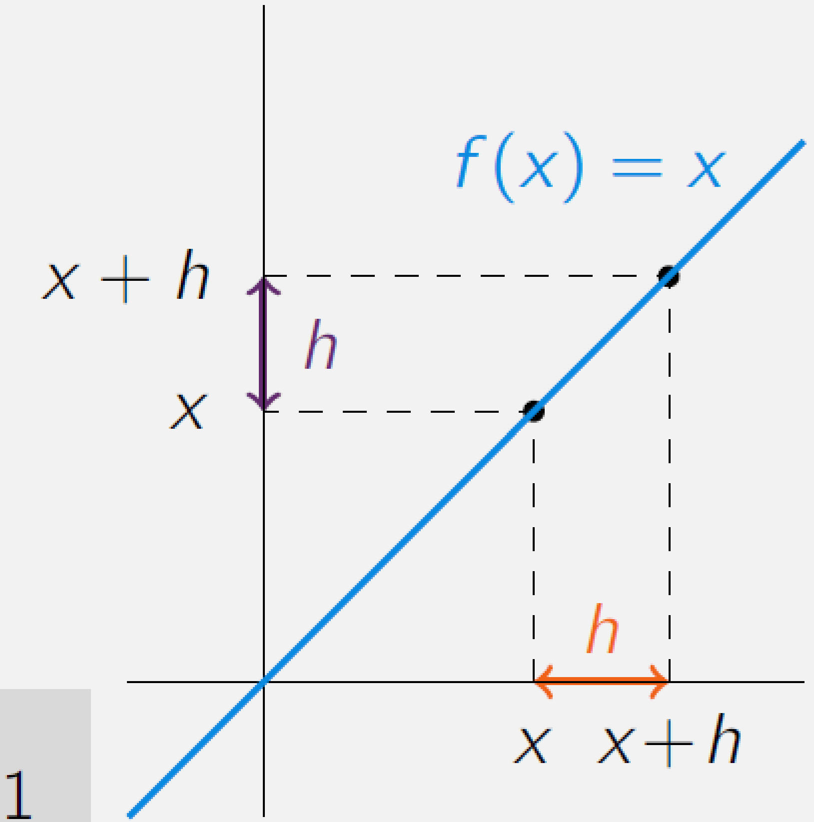
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 0 = 0$$

Standard Derivatives & Rules of Calculation

$$f(x) = x$$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h) - x}{h} = \frac{h}{h} = 1$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 1 = 1$$



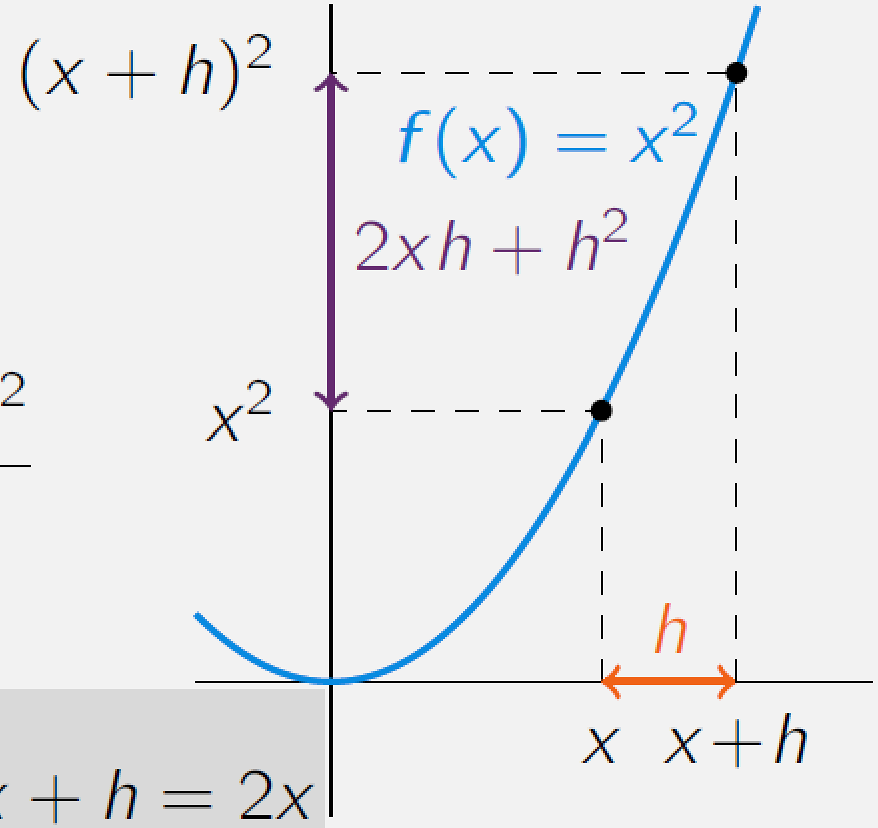
Standard Derivatives & Rules of Calculation

$$f(x) = x^2$$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h}$$

$$= \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= 2x + h$$



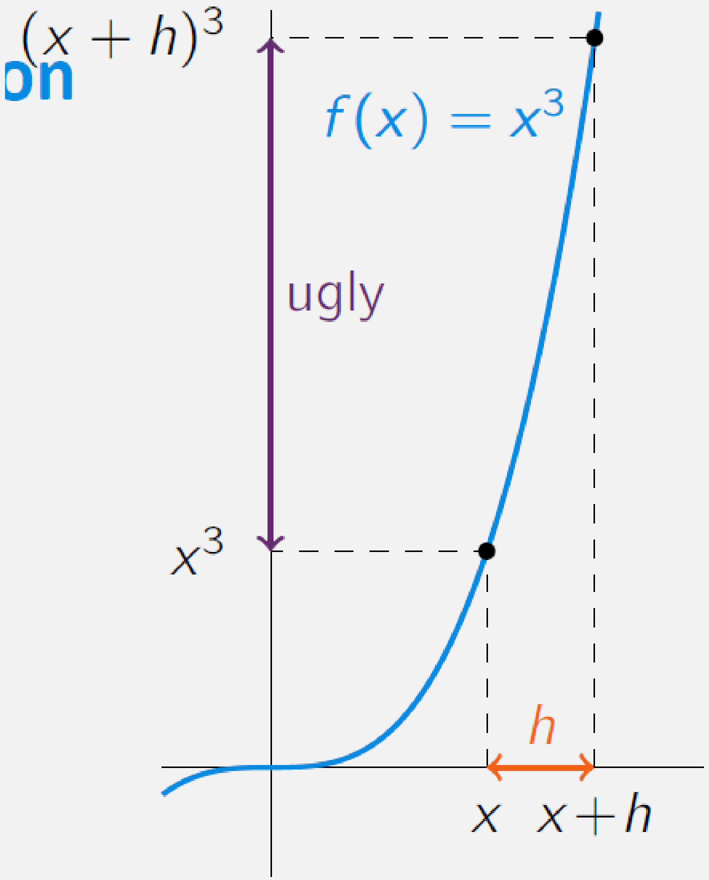
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 2x + h = 2x$$

Standard Derivatives & Rules of Calculation

$$f(x) = x^3$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\text{ugly}}{h}$$

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x \cdot x^2) \\ &= \left(\frac{d}{dx}x\right) \cdot x^2 + x\left(\frac{d}{dx}x^2\right) \\ &= 1 \cdot x^2 + x \cdot 2x \\ &= 3x^2 \end{aligned}$$



Standard Derivatives & Rules of Calculation

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}x^{\pi} = \pi x^{\pi-1}$$

Attention: the highlighted formula is valid only for superscripts that are constants

Standard Derivatives & Rules of Calculation

$$\frac{d}{dx}(2 \quad + 4x \quad + 5x^2 \quad + 7x^3 \quad - x^4 \ ) =$$

$$= 2 \cdot 0 + 4 \cdot 1 + 5 \cdot 2x + 7 \cdot 3x^2 - 4x^3$$

$$= 4 + 10x + 21x^2 - 4x^3$$

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Standard Derivatives & Rules of Calculation

Regarding the derivatives of functions of the type of a^x please remember the following:

Standard Derivatives & Rules of Calculation

S. No.	f (x)	f′ (x)
1	x^n	nx^{n-1}
2	Sin x	Cos x
3	Cos x	-Sin x
4	Tan x	Sec 2x
5	Cot x	-Cosec2x
6	Sec x	Sec x Tan x
7	Cosec x	-Cosec x Cot x
8	log x	1/x
9	a constant	Zero
10	e^x	e^x
11	a^x	$a^x \log a$
12	√x	1 / (2√x)

Basic Derivatives Rules

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$

Difference Rule: $\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x)+g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

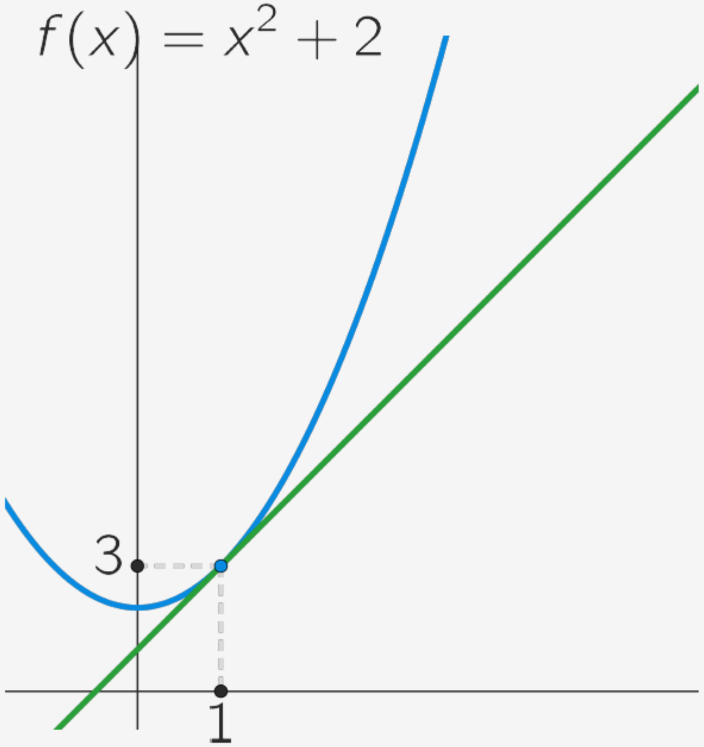
Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

Standard Derivatives & Rules of Calculation

- We know that **the derivative of a function at a point corresponds to the tangent line at this point of the function's graph**
- It is very important in order to realize the properties of a function, around a point, to derive the equation of this line, y=a*x+b

Standard Derivatives & Rules of Calculation

The tangent line – example

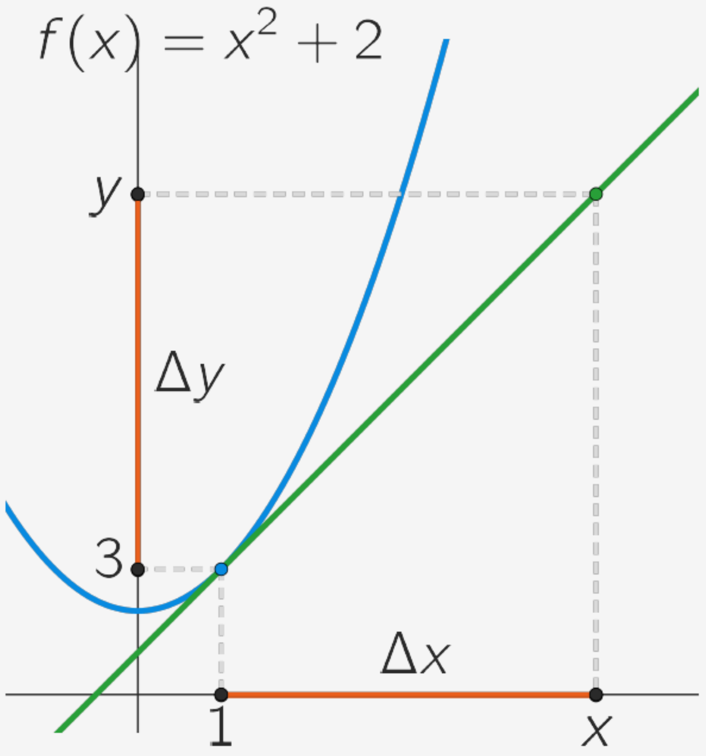


$$\text{Slope} = f'(1) = 2$$

$$f'(x) = \frac{d}{dx}(x^2 + 2) = 2x$$

Standard Derivatives & Rules of Calculation

The tangent line – example



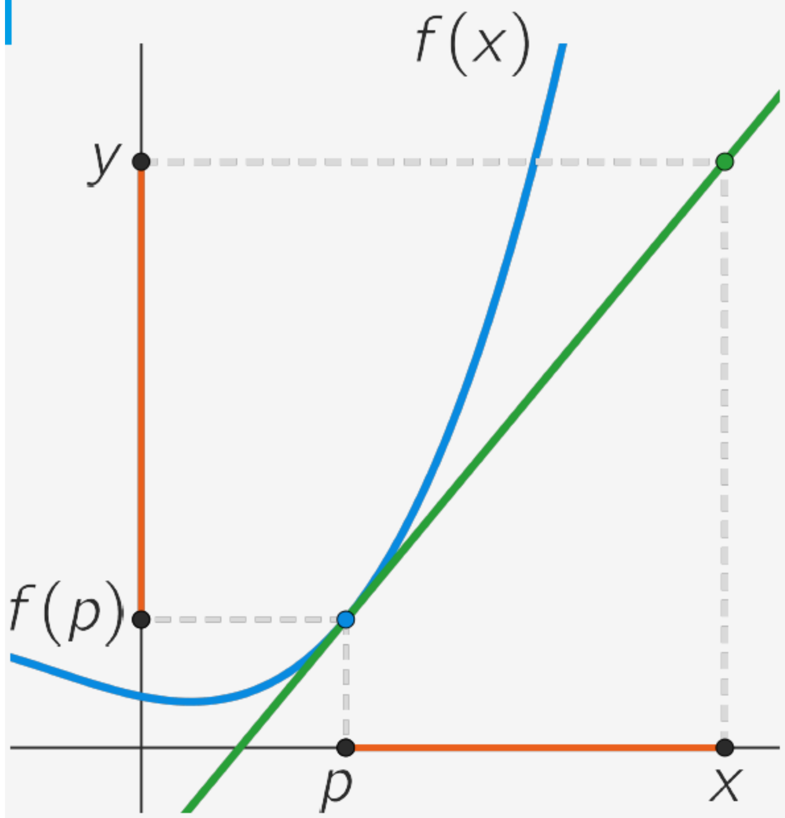
$$\text{Slope} = f'(1) = 2$$

$$2 = \frac{\Delta y}{\Delta x} = \frac{y-3}{x-1}$$

$$y - 3 = 2(x + 1)$$

Standard Derivatives & Rules of Calculation

The tangent line



Tangent line:

- slope $f'(p)$
- through $(p, f(p))$

$$f'(p) = \frac{y - f(p)}{x - p}$$

Equation tangent line:

$$y = f'(p)(x - p) + f(p)$$

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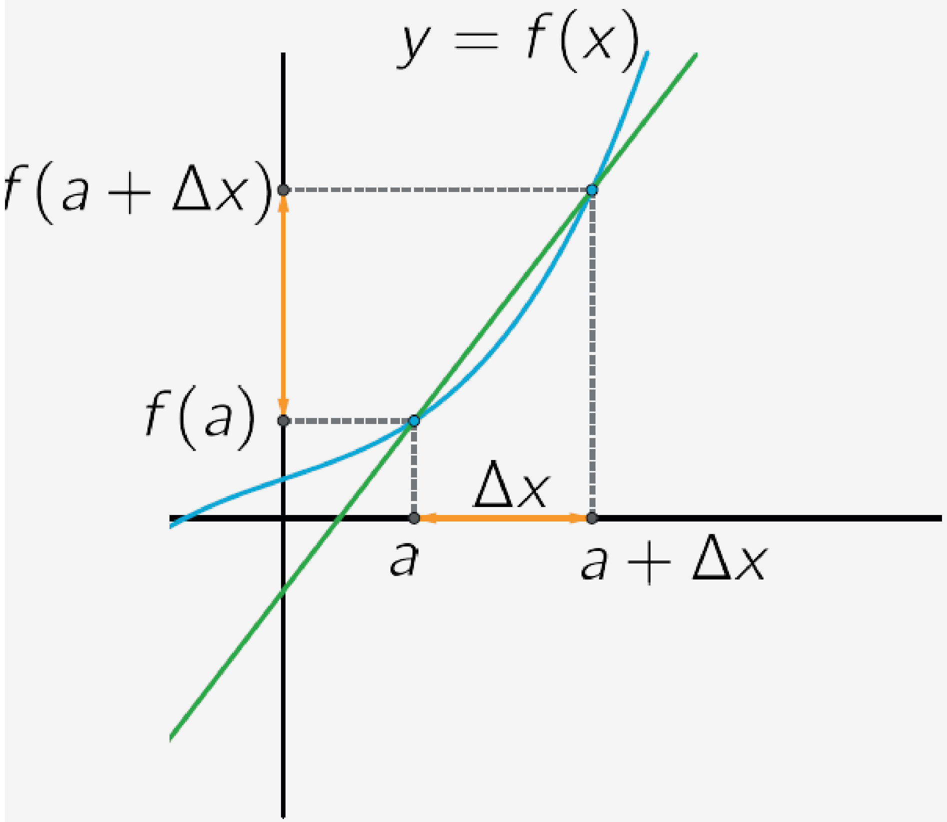
Standard Derivatives & Rules of Calculation

Non Differentiable Functions

- Can we differentiate any function? Can we differentiate any function at any point? **The answer to all of these questions is no!** What is the 'wrong' thing with the non differentiable functions?

- A function **is not differentiable at a point, when a tangent line passes through this point with a finite slope does not exist**

Standard Derivatives & Rules of Calculation



The function f is differentiable at a if

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

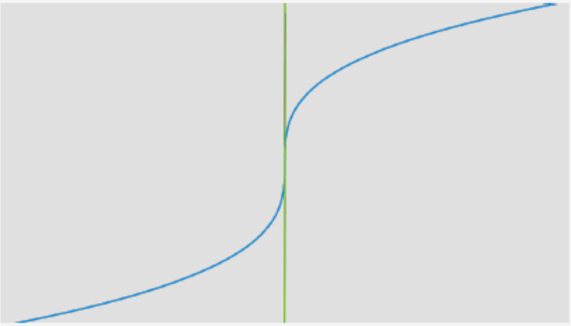
exists.

Standard Derivatives & Rules of Calculation

Nondifferentiable functions

Functions f that are not differentiable $x = a$:

- f is discontinuous at $x = a$.
- The graph of f has a kink at $x = a$.
- The graph of f has a vertical tangent line at $x = a$.



- More 'exotic' functions, e.g. $x \sin(\frac{1}{x})$.

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Standard Derivatives & Rules of Calculation

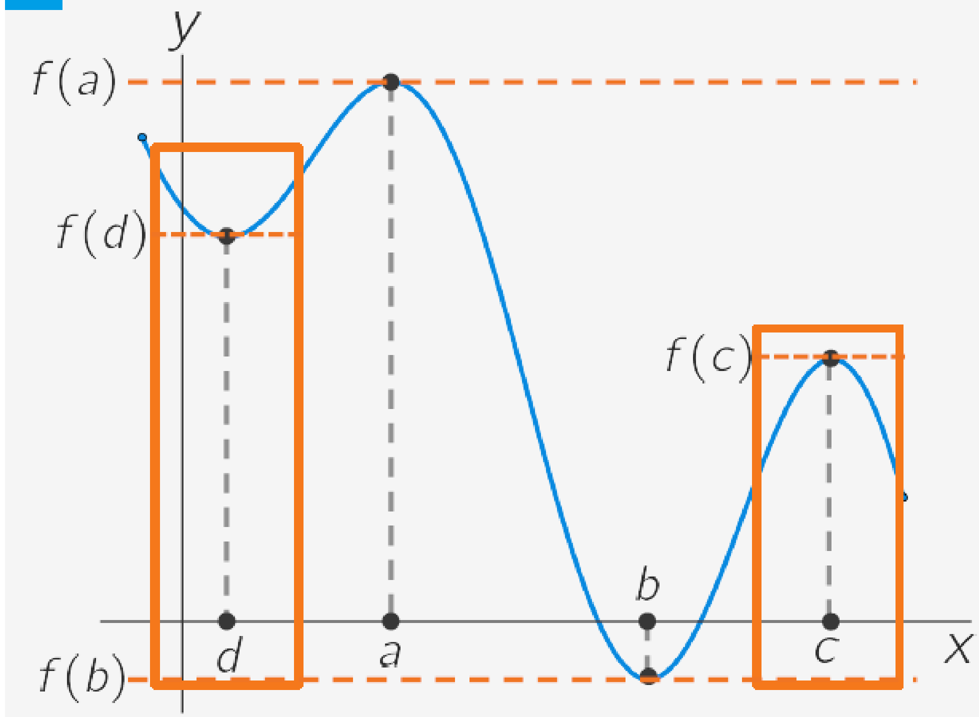
- An example of non-differentiable function is: $f(x) = x^{1/3}$ at zero. Can you explain why?
- Another example of a non-differentiable function is: f(x) = abs(x). Can you explain why?

Standard Derivatives & Rules of Calculation

- **For optimization** it is important **to be able to find the minima and maxima** of a function
- There are two types of maxima and minima within a graph: **the global ones** (the highest or the lowest value across the function's domain) or **the local ones** (the lowest or the highest one around a specific point)
- **Graph contains points that** can be called either (a) critical points (where the derivative is zero); (b) singular points (where the derivative is not defined); or (c) boundary points (where the derivative is not zero) – <u>Among these points we should check for maxima or minima points!!</u>

Standard Derivatives & Rules of Calculation

Minima and maxima



Global

maximum at $x = a$

minimum at $x = b$

Local

maximum at $x = c$

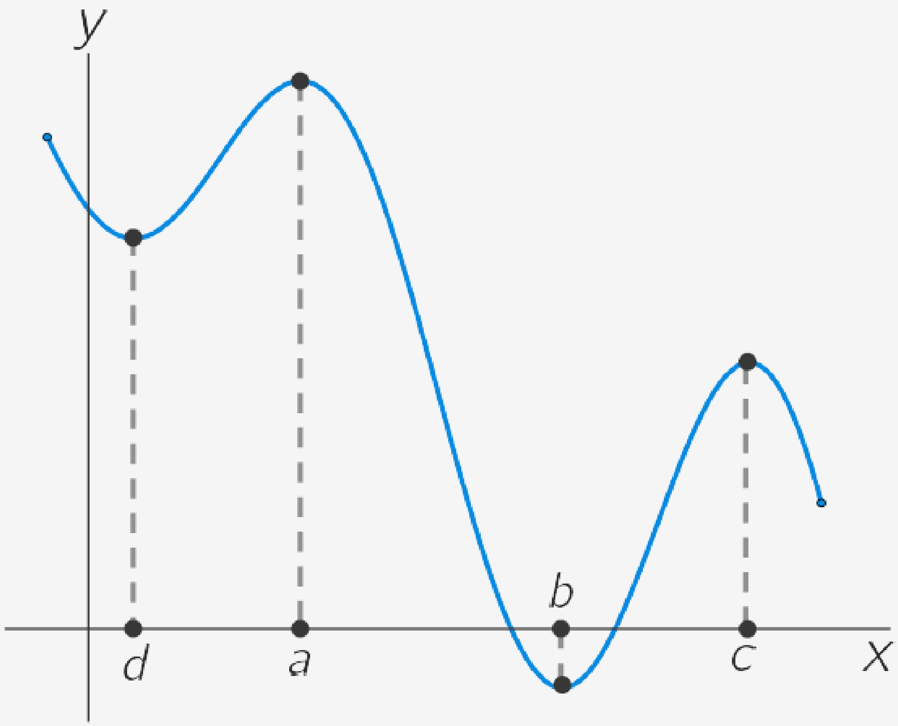
and $x = a$

minimum at $x = d$

and $x = b$

Standard Derivatives & Rules of Calculation

Minima and maxima



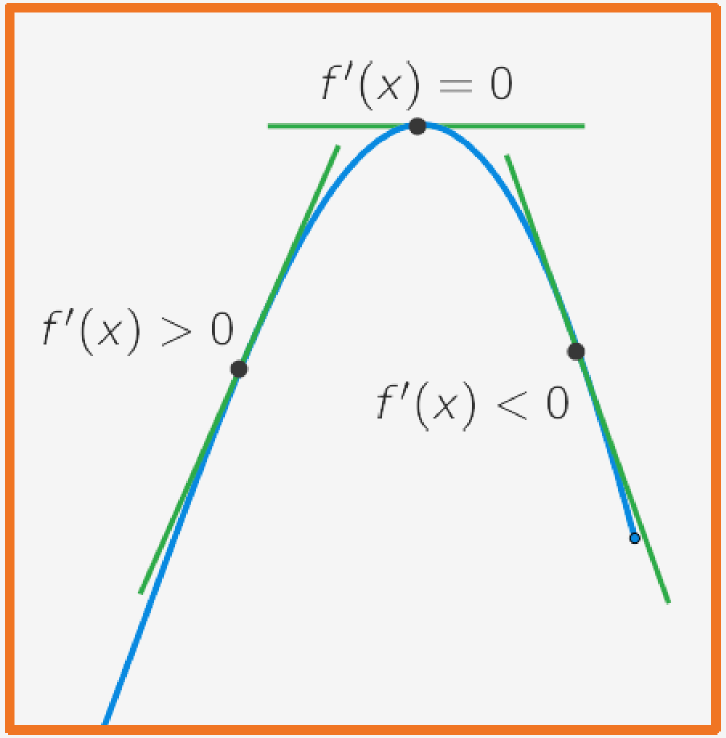
$f'(x) > 0 \Rightarrow$ increase

$f'(x) < 0 \Rightarrow$ decrease

at extrema: $f'(x) = 0$

Standard Derivatives & Rules of Calculation

Minima and maxima



$f'(x) > 0 \Rightarrow$ increase

$f'(x) < 0 \Rightarrow$ decrease

at extrema: $f'(x) = 0$

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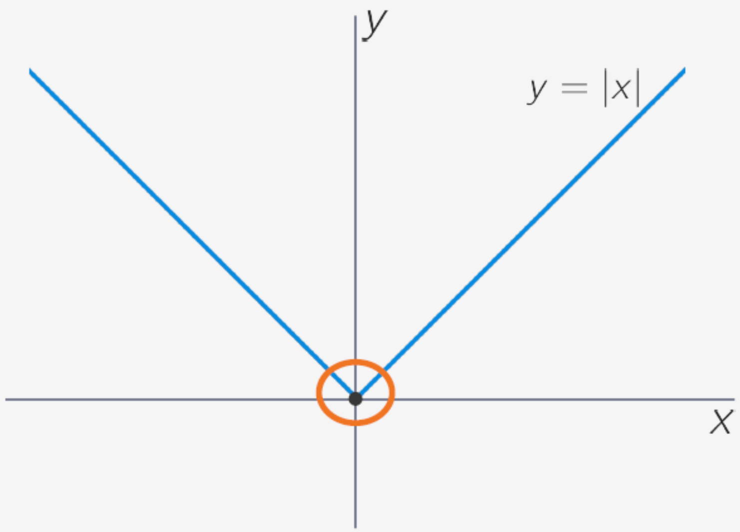
Standard Derivatives & Rules of Calculation

Non-differentiable functions



$f(x) = |x|$

minimum at $x = 0$

$f'(0)$ **does not exist!**

Boundary points

Minima at feet

f' exists

$f' \neq 0$ at feet

Boundary points

Standard Derivatives & Rules of Calculation

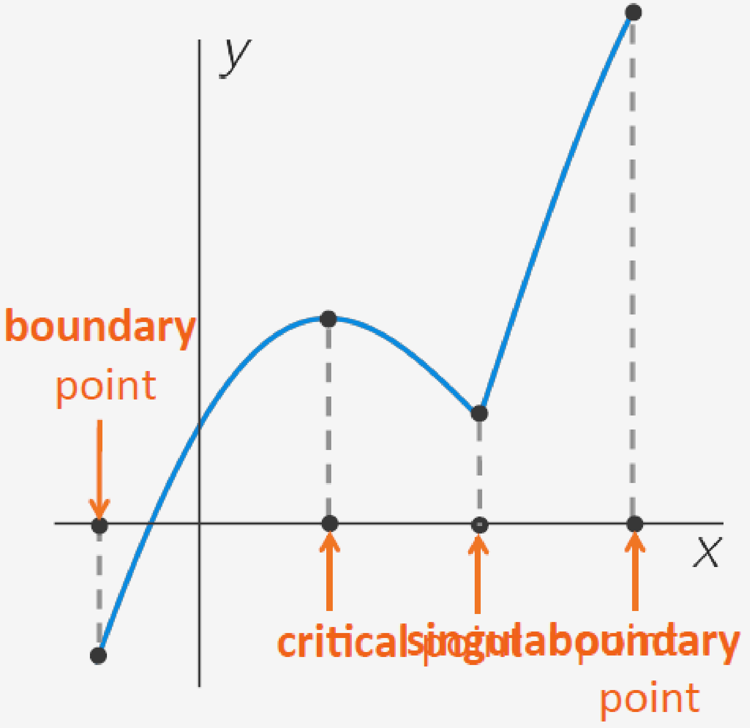
Finding minima and maxima

Given

- function f;
- point a such that $f(a)$ is local extremum;

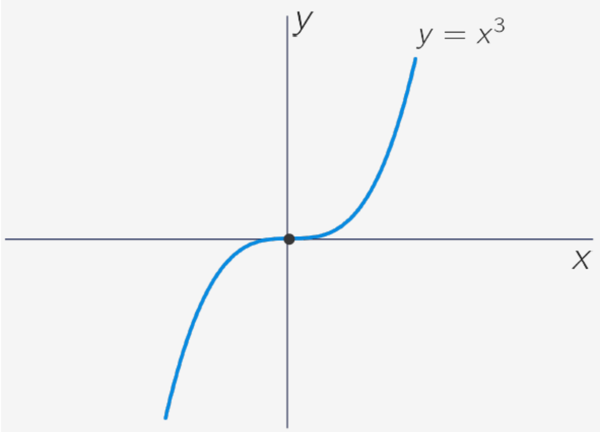
Then

- $f'(a) = 0$,
- or $f'(a)$ does not exist,
- or a is a boundary point.



Standard Derivatives & Rules of Calculation

Warning!



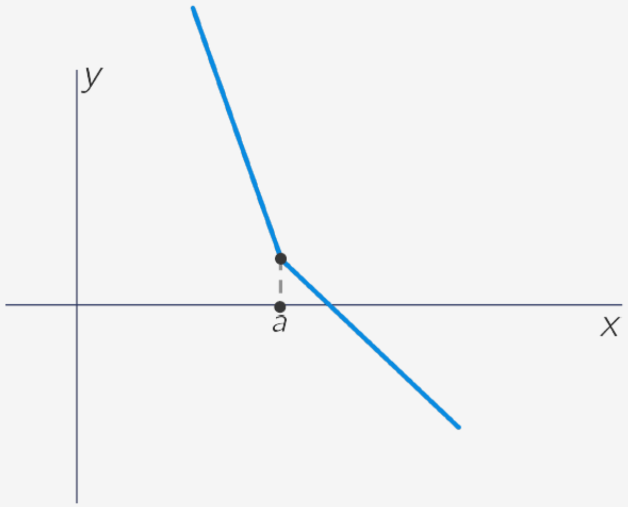
$f(x) = x^3$

$f'(x) = 3x^2$

$f'(0) = 0$

No extremum at $x = 0$

Warning!



Singular point $x = a$

No extremum at $x = a$

Standard Derivatives & Rules of Calculation

Finding local extrema

1. Find:
   - critical points: $f'(x) = 0$
   - singular points : $f'(x)$ does not exist
   - boundary points
2. Check at each at these points:
   - local minimum?
   - local maximum?
   - *neither*?

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Higher order differentiation

If *f* is a differentiable function and $f'(x) = \frac{df}{dx}$ its first derivative in respect to the variable *x*, then

the derivative of $f'(x)$ (if it exists) is denoted as

$$f''(x) = \frac{d^2x}{dx^2}$$

and is called <u>second derivative</u> of *f*.

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Higher order differentiation

The same way, the derivative of the second derivative (if it exists) is denoted as

$$f^{(3)}(x) = \frac{d^3x}{dx^3}$$

And is called the third derivative of *f*.

Continuing this process, from the *(v-1)*-th derivative of *f* we can derive the *v*-th derivative of *f*.

The *v*-th derivative is called **derivative of order *v***and is denoted as $f^{(v)}(x) = \frac{d^v x}{dx^v}$.

Higher order differentiation

Example:

Assume $f(x)=x^3-3x^2+2.$

Then

- $f'(x)=3x^2-6x$,
- $f''(x)=6x-6,$
- $f^{(3)}(x)=6$and
- $f^{(4)}(x)=0.$

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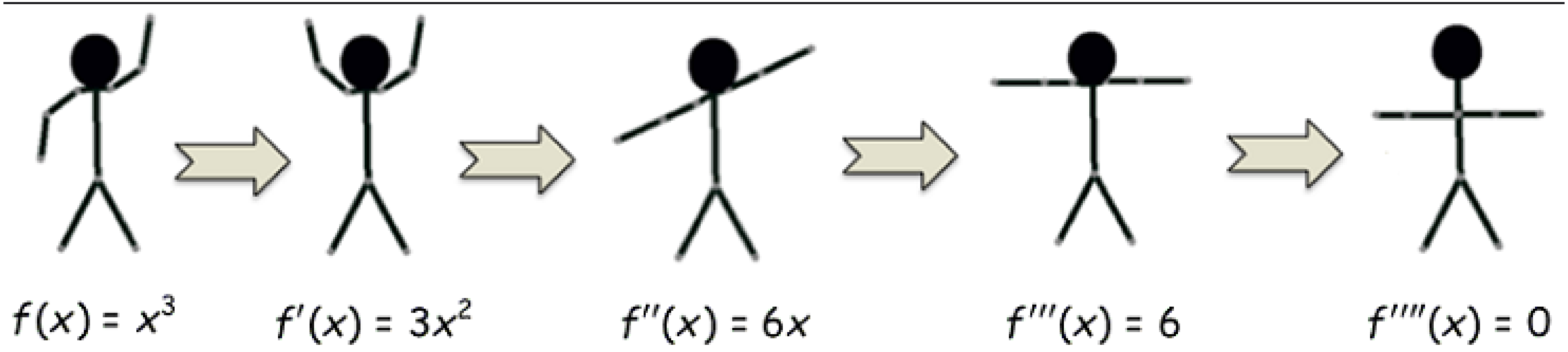
Higher order differentiation

The Derivatives Dance



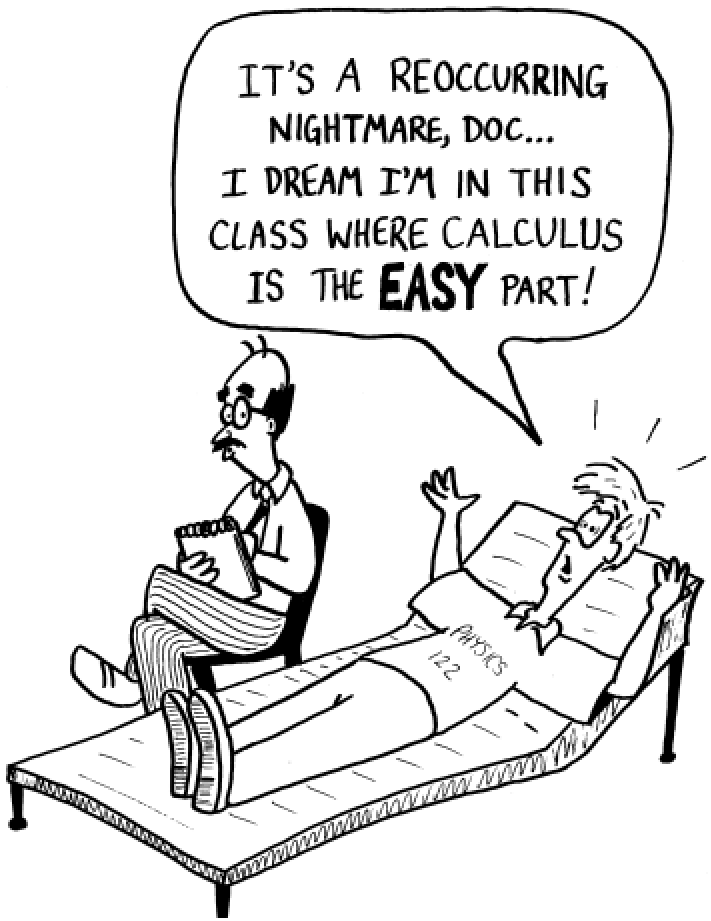
Differentiation

- If you want to learn more **about differentiation** please check the following video lectures:

    1. https://www.youtube.com/watch?v=xd703YLsLAY&feature=youtu.be

    2. https://www.youtube.com/watch?v=Rpum6FRM2UU&feature=youtu.be

Chapter Six: Integration


IT'S A REOCCURRING NIGHTMARE, DOC... I DREAM I'M IN THIS CLASS WHERE CALCULUS IS THE EASY PART!
PHYSICS 122

An Introduction

The aim of this chapter is: In this chapter we will learn **the definition of an integral**. This can be used to calculate distance from speed or speed from acceleration and to obtain the area under a curve for instance. Then we will **discuss standard integrals (definite - indefinite) and rules of integration**, followed by exercises to give you the chance to apply these rules.

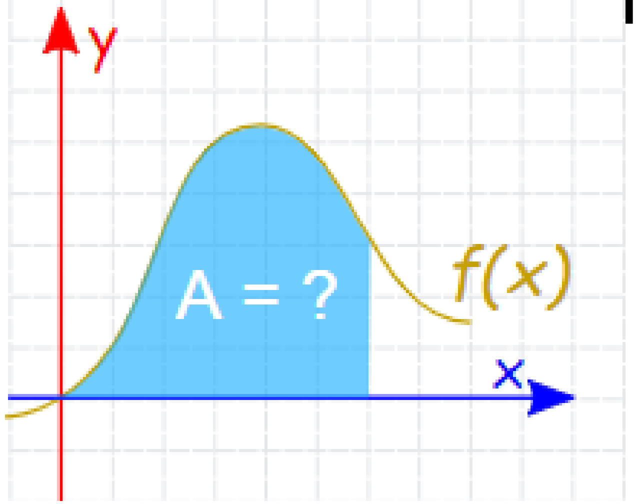
Definition of the Integration

- The integration is a mathematical action is defined as **the inverse operation of differentiation**
- The differentiation is related with the **area under** the curve of a function; it is also related to the **volume enclosed** by given surfaces; we need to integrate to also find the **length** of a curve.
- The process of integration is linked with the calculation of very important quantities. For example, in probability theory, integrals are used to determine the probability of some random variable falling within a certain range; integration is also used in physics, to find quantities like displacement, time, and velocity.

Definition of the Integration

Integration is a way of adding slices to find the whole.

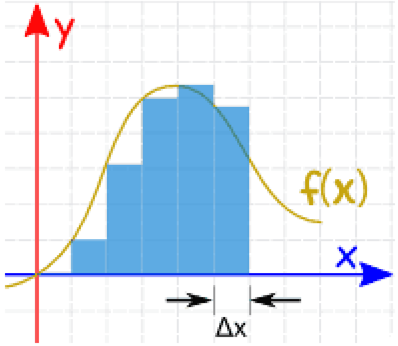
For example, let's say we want to calculate the area under the curve defined by the graph of *f(x)*:



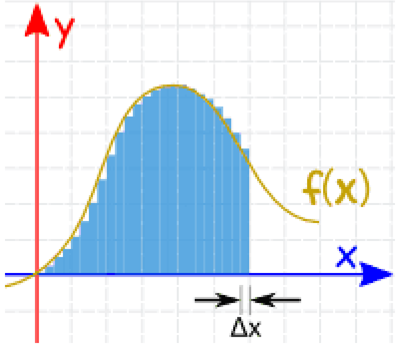
Definition of the Integration

We could calculate the function at a few points and **add up sli** **width Δx** like this (but the answer won't be very accurate):

A= $\sum f(x)\Delta x$



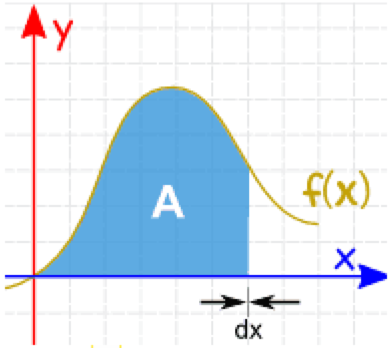
We can make **Δx** a lot smaller and **add up many small slices** (our answer will be much more accurate) **A=** $\sum f(x)\Delta x$



And as the **Δx** slices become smaller and smaller so as to **approach zero in width**, the answer approaches the **true answer**. We write **dx** t mean an infinitesimal **Δx.**

A= $\sum f(x)dx$

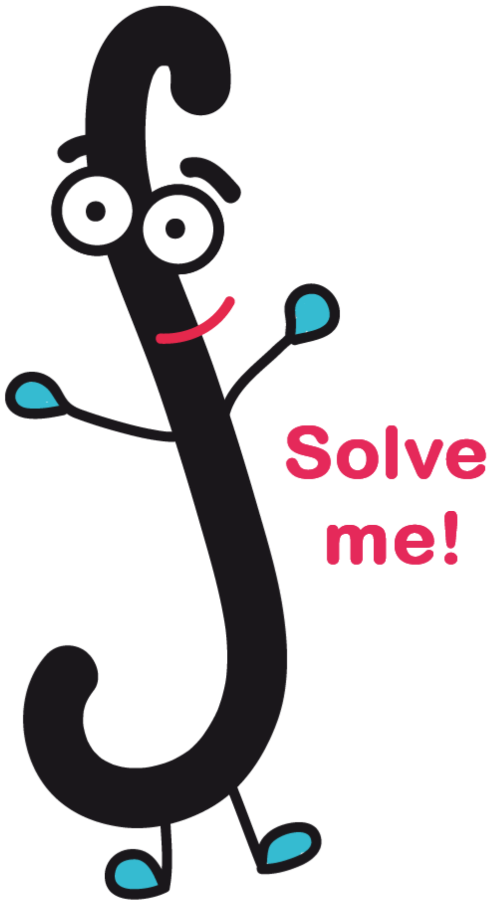
A= $\int f(x)dx$



Integration

$$\int f(x)dx$$

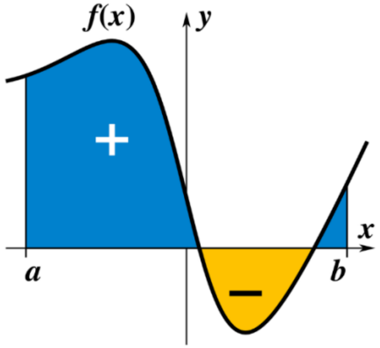
Symbol of integration
(can be seen as a “stretched **Σ**”)



Definite Integral

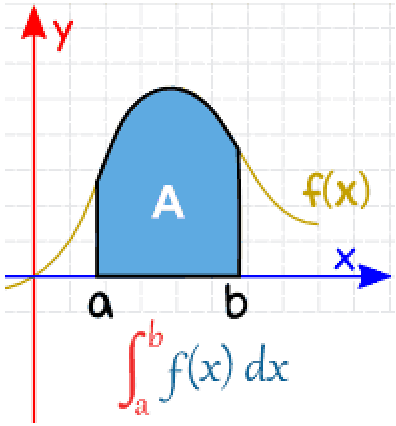
If f is a function defined in an interval $[a, b]$ then

the definite integral of $f(x)$ in respect to x from the point *a* to the point *b* is the area enclosed by the x-axis and the curve $y = f(x)$.



If the curve $y = f(x)$ is above the *x* axis we say that the area is positive, while if the curve $y = f(x)$ is below the *x* axis we say that the area is negative.

Definite Integral



The definite integral of $f(x)$ from *a* to *b* is written as $\int_a^b f(x)dx$

For the definite integrals the following rules exist:

If the right and the left limit points of the integral coincide there is no enclosed area and the definite integral equals zero, i.e. $\int_a^a f(x)\,dx = 0$.

If the right and the left limit points of the integral are swapped, the sign of the integral is changed, i. e. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

If $c \in [a, b]$then

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Definite Integral

If $\lambda \in \mathbb{C}$then $\int_a^b [\lambda\, f(x)]\, dx = \lambda \int_a^b f(x)\, dx$

If $g$is another function defined in $[a, b]$, then

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

$$\int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

If $\forall\ x \in [a, b]$there is $f(x) \leq g(x)$ then $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$

The antiderivative

If f is a function defined in the interval $[a, b]$ then

The function $F(x)$ is called primitive function or antiderivative of $f(x)$ if

$$F'(x) = f(x), \qquad \forall x \in [a, b].$$

It is obvious that if $F(x)$ is an antiderivative of $f(x)$, then
then function $F(x) + c$, where c is a constant,
will also be an antiderivative of $f(x)$, because $[F(x) + c]' = F'(x)$.

The antiderivative - example

The antiderivative of $f(x) = 2x$is the function $F(x)$=x^2, because

$$(x^2)' = 2x$$

The same is valid for all functions $F(x)$=$x^2 + c$, where c is a constant, because $[x^2 + c]' = 2x$

Antiderivatives of basic functions

$f(x)$	$F(x)$	$f(x)$	$F(x)$
0	c	$\frac{1}{x}$	$\ln\lvert x\rvert + c$
a	$ax + c$	$a^x, \quad a > 0$	$\frac{a^x}{ln(a)} + c$
$x^a, \qquad a \neq -1$	$\frac{1}{a+1} x^{a+1} + c$	e^{ax}	$\frac{1}{a} e^{ax} + c$
$sin(ax)$	$-\frac{1}{a} cos(ax) + c$	$sinh(ax)$	$\frac{1}{a} cosh(ax) + c$
$cos(ax)$	$\frac{1}{a} sin(ax) + c$	$cosh(ax)$	$\frac{1}{a} sinh(ax) + c$

Indefinite Integral

The indefinite integral of the function $f(x)$ in respect to the variable x is the set of all the antiderivatives of $f(x)$ and is the integral $\int f(x)\,dx$.

Since $F(x)$ is the antiderivative of $f(x)$, there is $F'(x) = f(x)$, so there is

$$\int f(x)dx = \int F'(x)dx = F(x) + c$$

...In other words the integration is the inverse process of differentiation!

So integrating actually means, finding the antiderivative!!!

Indefinite Integral

So integrating actually means, finding the antiderivative!!!

Example

$$\int 2x\,dx = x^2 + c$$

The fundamental theorem of calculus

If $f(x)$ is a continuous function defined in the interval $[a, b]$ and $F(x)$ is its antiderivative then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This theorem allows us to calculate the value of definite integrals when we know how to calculate the indefinite integral (when we know the antiderivative)

The fundamental theorem of calculus

Example

To find the value of $\int_1^2 2x \ dx$ we observe that the antiderivative of the function inside the integral $f(x) = 2x$ is the function $F(x)=x^2 + c$ and so there is:

$$\int_1^2 2x\, dx = (2^2 + c) - (1^2 + c) = 3$$

We can simplify our calculations here since the constant c does not play any role:

$$\int_1^2 2x\, dx = x^2 \Big|_1^2 = 2^2 - 1^2 = 3.$$

Integration rules

Integration by parts

From the differentiation rules we know tha $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

so $f'(x)g(x) = [f(x)g(x)]' - f(x)g'(x).$

Integrating both parts of the equation we g $\int f'(x)g(x)\,dx = \int [f(x)g(x)]'dx - \int f(x)g'(x)dx$

which means that $\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)dx.$

Integration by parts

Example

To find the value of the definite integral $\int_0^{\pi} xsin(x)\,dx$ we observe that

$xsin(x) = x[-\cos(x)]'$.

Therefore,

$$\int_0^{\pi} xsin(x)\,dx = \int_0^{\pi} x[-\cos(x)]'\,dx = -\int_0^{\pi} x[\cos(x)]'\,dx$$

Similarly,

$$\int_0^{\pi} x[\cos(x)]'\,dx = xcos(x)\Big|_0^{\pi} - \int_0^{\pi} x'\cos(x)\;dx = [\pi cos(\pi) - 0] - \int_0^{\pi} \cos(x)\;dx =$$

$$= -\pi - sin(x)\Big|_0^{\pi} = -\pi - 0 = -\pi.$$

Άρα, $\int_0^{\pi} xsin(x)\;dx = \pi$.

Integration rules

Integration by substitution (change of variable)

If we have to calculate an integral of the form

$$I = \int f(g(x))g'(x)dx$$

the steps we follow are these:

1. we make the change of variable $g(x) = u$. Since $du = g'(x)dx$, after the substitution we have
$$I = \int f(u)du$$
2. we integrate in respect to u
3. when we finish we substitute again the variable u with $g(x)$.

Integration by substitution

Example

The indefinite integral

$$\int \frac{x}{\sqrt{1-x^2}}dx$$

can be calculated as follows:

1. we observe that if we put $g(x) = 1 - x^2$ then $g'(x) = -2x$.
   and therefore

$$\frac{x}{\sqrt{1-x^2}} = -\frac{1}{2}\frac{g'(x)}{\sqrt{g(x)}}$$

   so

$$\int \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{g'(x)}{\sqrt{g(x)}}dx$$

2. we substitute $g(x) = u$ and $g'(x)dx = du$. Then we have:

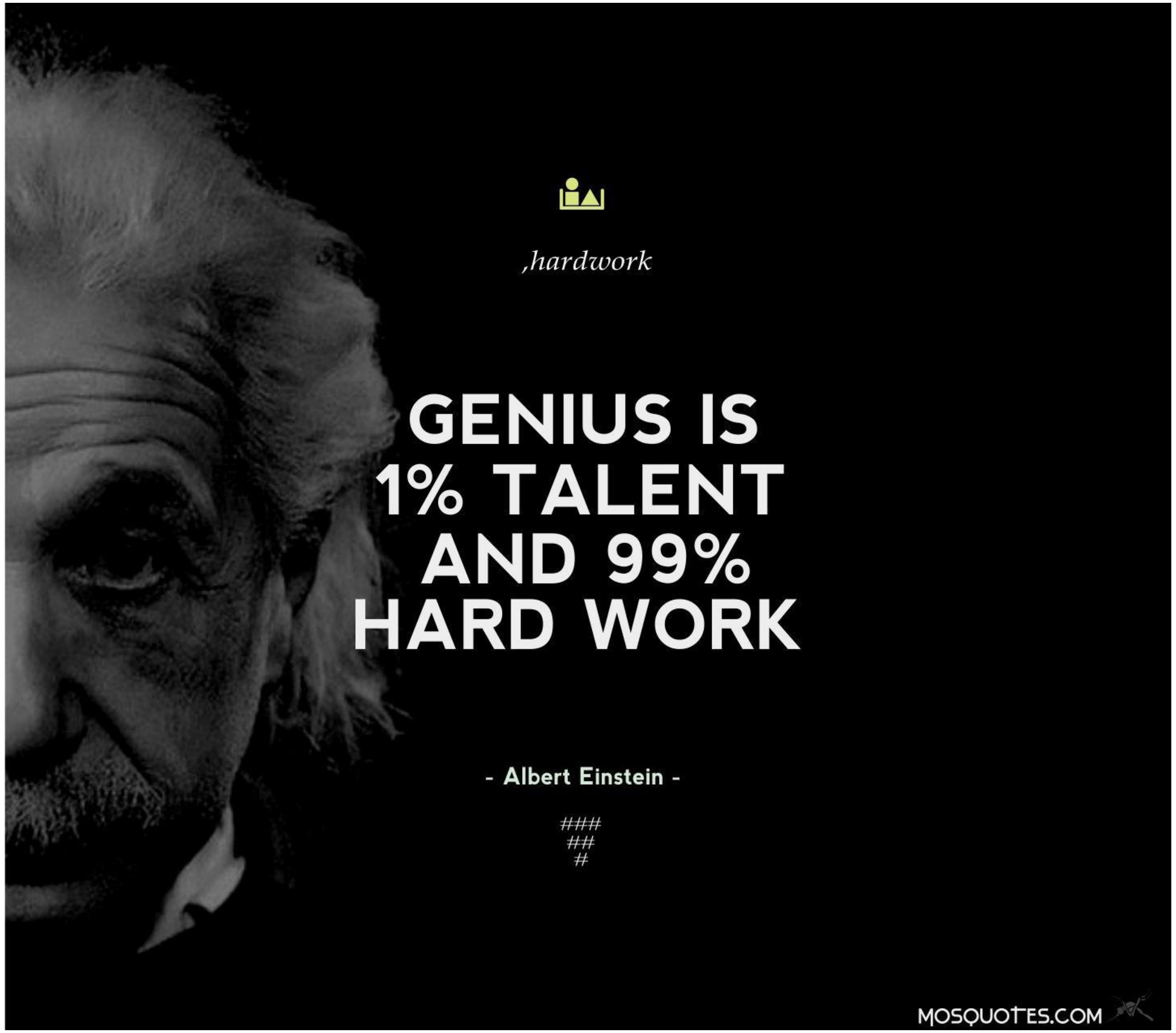
$$\int \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{1}{\sqrt{u}}du$$

3. we integrate

$$-\frac{1}{2}\int \frac{1}{\sqrt{u}}du = -\frac{1}{2}\int u^{-\frac{1}{2}}du = \left(-\frac{1}{2}\right)\left(\frac{1}{1-\frac{1}{2}}u^{1-\frac{1}{2}}\right) + c = -u^{\frac{1}{2}} + c = -\sqrt{u} + c$$

4. we do not forget to replace u with our initial $g(x) = 1 - x^2$:

$$\int \frac{x}{\sqrt{1-x^2}}dx = -\sqrt{u} + c = -\sqrt{1-x^2} + c$$


,hardwork
GENIUS IS 1% TALENT AND 99% HARD WORK
- Albert Einstein -
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